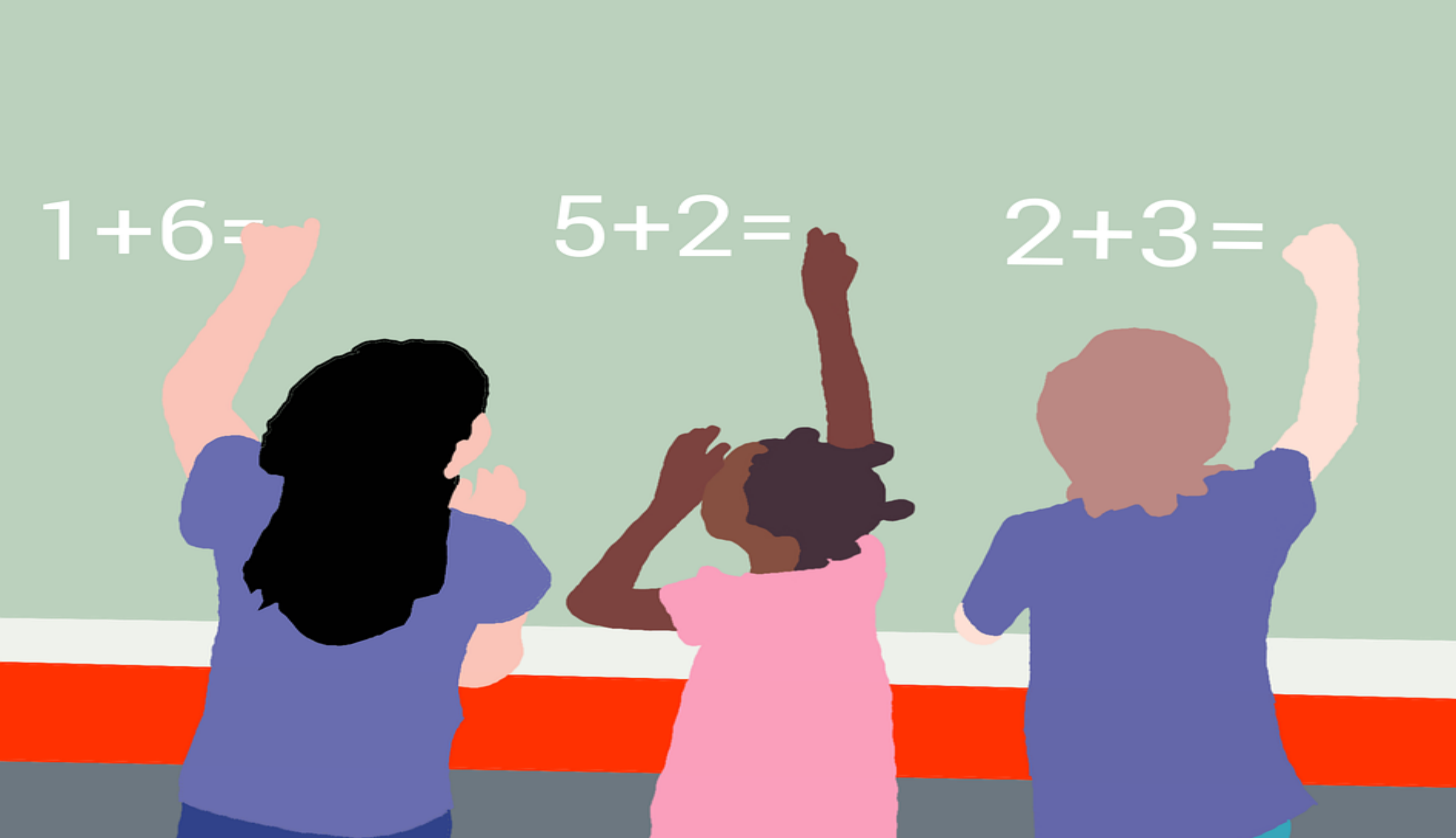
TeachME PROFESSIONAL DEVELOPMENT
CEU's for Teachers and Educators

Improving Mathematical Problem Solving-2018 Update



Institute of Education Sciences Levels of Evidence for Practice Guides

This section provides information about the role of evidence in Institute of Education Sciences' (IES) What Works Clearinghouse (WWC) practice guides. It describes how practice guide panels determine the level of evidence for each recommendation and explains the criteria for each of the three levels of evidence (strong evidence, moderate evidence, and minimal evidence).

The level of evidence assigned to each recommendation in this practice guide represents the panel's judgment of the quality of the existing research to support a claim that, when these practices were implemented in past research, positive effects were observed on student outcomes. After careful review of the studies supporting each recommendation, panelists determine the level of evidence for each recommendation using the criteria in Table 1. The panel first considers the relevance of individual studies to the recommendation and then discusses the entire evidence base, taking the following into consideration:

- the number of studies
- the design of the studies
- the quality of the studies
- whether the studies represent the range of participants and settings on which the recommendation is focused
- whether findings from the studies can be attributed to the recommended practice
- whether findings in the studies are consistently positive

A rating of *strong evidence* refers to consistent evidence that the recommended strategies, programs, or practices improve student outcomes for a wide population of students.[1] In other words, there is strong causal and generalizable evidence.

A rating of *moderate evidence* refers either to evidence from studies that allow strong causal conclusions but cannot be generalized with assurance to the population on which a recommendation is focused (perhaps because the findings have not been widely replicated) or to evidence from studies that are generalizable but have some causal ambiguity. It also might be that the studies that exist do not specifically examine the outcomes of interest in the practice guide, although they may be related.

A rating of *minimal evidence* suggests that the panel cannot point to a body of research that demonstrates the practice's positive effect on student achievement. In some cases, this simply means that the recommended practices would be difficult to study in a rigorous, experimental fashion;[2] in other cases, it means that researchers have not yet studied this practice, or that there is weak or conflicting evidence of effectiveness. A minimal evidence rating does not indicate that the recommendation is any less important than other recommendations with a strong evidence or moderate evidence rating.

In developing the levels of evidence, the panel considers each of the criteria in Table 1. The level of evidence rating is determined as the lowest rating achieved for any individual criterion. Thus, for a recommendation to get a strong rating, the research must be rated as strong on each criterion. If at least one criterion receives a rating of moderate and none receive a rating of minimal, then the level of evidence is determined to be moderate. If one or more criteria receive a rating of minimal, then the level of evidence is determined to be minimal.

Table 1. Institute of Education Sciences levels of evidence for practice guides

Criteria	STRONG Evidence Base	MODERATE Evidence Base	MINIMAL Evidence Base
Validity	High internal validity (high-quality causal designs). Studies must meet WWC standards with or without reservations.[3] **AND** High external validity (requires multiple studies with high-quality causal designs that represent the population on which the recommendation is focused). Studies must meet WWC standards with or without reservations.	High internal validity but moderate external validity (i.e., studies that support strong causal conclusions but generalization is uncertain). **OR** High external validity but moderate internal validity (i.e., studies that support the generality of a relation but the causality is uncertain).[4]	The research may include evidence from studies that do not meet the criteria for moderate or strong evidence (e.g., case studies, qualitative research).
Effects on relevant outcomes	Consistent positive effects without contradictory evidence (i.e., no statistically significant negative effects) in studies with high internal validity.	A preponderance of evidence of positive effects. Contradictory evidence (i.e., statistically significant negative effects) must be discussed by the panel and considered with regard to relevance to the scope of the guide and intensity of the recommendation as a component of the intervention evaluated.	There may be weak or contradictory evidence of effects.
Relevance to scope	Direct relevance to scope (i.e., ecological validity)—relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated.	Relevance to scope (ecological validity) may vary, including relevant context (e.g., classroom vs. laboratory), sample (e.g., age and characteristics), and outcomes evaluated. At least some research is directly relevant to scope (but the research that is relevant to scope does not qualify as strong with respect to validity).	The research may be out of the scope of the practice guide.
Relationship between research and recommendations	Direct test of the recommendation in the studies or the recommendation is a major component of the intervention tested in the studies.	Intensity of the recommendation as a component of the interventions evaluated in the studies may vary.	Studies for which the intensity of the recommendation as a component of the interventions evaluated in the studies is low; and/or the recommendation reflects expert opinion based on reasonable extrapolations from research.

Table 1. Institute of Education Sciences levels of evidence for practice guides *(continued)*

Criteria	STRONG Evidence Base	MODERATE Evidence Base	MINIMAL Evidence Base
Panel confidence	Panel has a high degree of confidence that this practice is effective.	The panel determines that the research does not rise to the level of strong but is more compelling than a minimal level of evidence. Panel may not be confident about whether the research has effectively controlled for other explanations or whether the practice would be effective in most or all contexts.	In the panel's opinion, the recommendation must be addressed as part of the practice guide; however, the panel cannot point to a body of research that rises to the level of moderate or strong.
Role of expert opinion	Not applicable	Not applicable	Expert opinion based on defensible interpretations of theory (theories). (In some cases, this simply means that the recommended practices would be difficult to study in a rigorous, experimental fashion; in other cases, it means that researchers have not yet studied this practice.)
When assessment is the focus of the recommendation	For assessments, meets the standards of *The Standards for Educational and Psychological Testing.*[5]	For assessments, evidence of reliability that meets *The Standards for Educational and Psychological Testing* but with evidence of validity from samples not adequately representative of the population on which the recommendation is focused.	Not applicable

The panel relied on WWC evidence standards to assess the quality of evidence supporting educational programs and practices. WWC evaluates evidence for the causal validity of instructional programs and practices according to WWC standards. Information about these standards is available at http://ies.ed.gov/ncee/wwc/DocumentSum.aspx?sid=19. Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages.

Introduction to the *Improving Mathematical Problem Solving in Grades 4 Through 8* Practice Guide

This section outlines the importance of improving mathematical problem solving for students in grades 4 through 8 and explains key parameters considered by the panel in developing the practice guide. It also summarizes the recommendations for readers and concludes with a discussion of the research supporting the practice guide.

Students who develop proficiency in mathematical problem solving early are better prepared for advanced mathematics and other complex problem-solving tasks.[6] Unfortunately, when compared with students in other countries, students in the U.S. are less prepared to solve mathematical problems.[7] For example, recent Trends in International Mathematics and Science Study (TIMSS) data suggest that, when compared to other industrialized countries such as the Netherlands, China, and Latvia, U.S. 4th-graders rank tenth and 8th-graders rank seventh out of 41 countries in problem solving.[8]

Problem solving involves reasoning and analysis, argument construction, and the development of innovative strategies. These abilities are used not only in advanced mathematics topics—such as algebra, geometry and calculus—but also throughout the entire mathematics curriculum beginning in kindergarten, as well as in subjects such as science. Moreover, these skills have a direct impact on students' achievement scores, as many state and national standardized assessments and college entrance exams include problem solving.[9]

Traditional textbooks[10] often do not provide students rich experiences in problem solving.[11] Textbooks are dominated by sets of problems that are not cognitively demanding, particularly when assigned as independent seatwork or homework, and teachers often review the answers quickly without discussing what strategies students used to solve the problems or whether the solutions can be justified.[12] The lack of guidance in textbooks is not surprising, given that state and district standards are often less clear in their guidelines for process skills, such as problem solving, than they are in their wording of grade-level content standards.[13]

The goal of this practice guide is to give teachers and administrators recommendations for improving mathematical problem-solving skills, regardless of which curriculum is used. The guide offers five recommendations that provide teachers with a coherent approach for regularly incorporating problem solving into their classroom instruction to achieve this end. It presents evidence-based suggestions for putting each recommendation into practice and describes roadblocks that may be encountered, as well as possible solutions.

Scope of the practice guide

Audience and grade level. The need for effective problem-solving instruction is particularly critical in grades 4 through 8, when the mathematics concepts taught become more complicated and when various forms of assessments—from class tests to state and national assessments—begin incorporating problem-solving activities. In this guide, the panel provides teachers with five recommendations for instructional practices that improve students' problem-solving ability. Math coaches and other administrators also may find this guide helpful as they prepare teachers to use these practices in their classrooms. Curriculum developers may find the guide useful in making design decisions, and researchers may find opportunities to extend or explore variations in the evidence base.

Content. The literature reviewed for this guide was restricted to mathematical problem-solving topics typically taught in grades 4 through 8. The panelists reviewed a number of definitions of *problem solving* as part of the process of creating this guide, but a single, prevalent definition of *problem solving* was not identified. This is understandable, given the different contexts in which the term *problem solving* is used in mathematics. Some definitions are exceedingly broad and applied to a general level of problem solving that goes beyond mathematics into everyday human affairs. For example, *problem solving* is often defined as the "movement from a given state to a goal state with no obvious way or method for getting from one to the other."[14] This kind of definition underscores the non-routine nature of problem solving and the fact that it is not the execution of memorized rules or shortcuts, such as using key words, to solve math word problems.

More contemporary definitions of *problem solving* focus on communication, reasoning, and multiple solutions. In addition to the non-routine nature of the process, this kind of mathematical problem solving is portrayed as the opportunity to engage in mathematics and derive a reasonable way or ways to solve the problem.[15] In light of the long-standing historical variations and disputes over definitions of *problem solving*, the panel ultimately decided that it was not in their purview to resolve this issue. The panel defined the characteristics of problem solving that applied to this guide as follows:

- First, students can learn mathematical problem solving; it is neither an innate talent nor happenstance that creates skilled problem solvers.
- Second, mathematical problem solving is relative to the individual. What is challenging or non-routine for one student may be comparatively straightforward for a more advanced student.
- Third, mathematical problem solving need not be treated like just another topic in the pacing guide; instead, it can serve to support and enrich the learning of mathematics concepts and notation.
- Fourth, often more than one strategy can be used to solve a problem. Learning multiple strategies may help students see different ideas and approaches for solving problems and may enable students to think more flexibly when presented with a problem that does not have an obvious solution.

Problem solving includes more than working *word problems*. While word problems have been the mainstay of mathematics textbooks for decades, they are only one type of math problem. Other types of math problems appropriate to grades 4 through 8, such as algebraic and visual-spatial problems (e.g., "How many squares are there on a checkerboard?"), are addressed in this guide. The panel excluded whole number addition and subtraction, which are typically taught in kindergarten through grade 3, as well as advanced algebra and advanced geometry, which are typically taught in high school.

When developing recommendations, the panel incorporated several effective instructional practices, including explicit teacher modeling and instruction, guided questions, and efforts to engage students in conversations about their thinking and problem solving. The panel believes it is important to include the variety of ways problem solving can be taught.

There are several limitations to the scope of this guide. The literature reviewed for this guide was limited to studies pertaining to mathematical problem solving; therefore, it did not include cognitive or psychological dimensions of problem solving that fell outside of this topic area.[16] While the panel considered studies that included students with disabilities and students who were learning English, this guide does not address specific instructional practices for these groups. Instead, this guide is intended for use by all teachers, including general education, special education teachers, and teachers of English learners, of mathematics in grades 4 through 8.

Summary of the recommendations

The five recommendations in this guide can be used independently or in combination to help teachers engage students in problem solving on a regular basis. To facilitate using the recommendations in combination, the panel provided a discussion of how the recommendations can be combined in the lesson-planning process. This discussion is presented in the conclusion section of the guide.

Recommendation 1 explains how teachers should incorporate problem-solving activities into daily instruction, instead of saving them for independent seatwork or homework. The panel stresses that teachers must consider their unit goals and their students' background and interests when preparing problem-solving lessons.

Recommendation 2 underscores the importance of thinking through or reflecting on the problem-solving process. Thinking through the answers to questions such as "What is the question asking me to do?" and "Why did these steps in solving the problem work or not work?" will help students master multistep or complex problems.

Recommendations 3, 4, and 5 focus on specific ways to teach problem solving.

Recommendation 3 covers instruction in visual representations, such as tables, graphs, and diagrams. Well-chosen visual representations help students focus on what is central to many mathematical problems: the relationship between quantities.

Recommendation 4 encourages teachers to teach multiple strategies that can be used to solve a problem. Sharing, comparing, and discussing strategies afford students the opportunity to communicate their thinking and, by listening to others, become increasingly flexible in the way they approach and solve problems. Too often students become wedded to just one approach and then flounder when it does not work on a different or more challenging problem.

Recommendation 5 encourages teachers to help students recognize and articulate mathematical concepts and notation during problem-solving activities. The key here is for teachers to remember that students' problem solving will improve when students understand the formal mathematics at the heart of each problem.

Of the five recommendations the panel shares in this guide, the panel chose to present the recommendation (Recommendation 1) that provides guidance for preparing problem-solving activities first. Even though the level of evidence supporting this recommendation is not strong, the panel believes teachers should plan before undertaking these activities. The first two recommendations can be used regularly when preparing and implementing problem-solving lessons; in contrast, the panel does not think recommendations 3 through 5 must be used in *every* lesson. Instead, teachers should choose the recommendations that align best with their goals for a given lesson and its problems. For example, there are occasions when visual representations are not used as part of problem-solving instruction, such as when students solve an equation by considering which values of the variable will make both sides equal.

Use of research

The evidence used to create and support the recommendations in this practice guide ranges from rigorous experimental studies to expert reviews of practices and strategies in mathematics education; however, the evidence ratings are based solely on high-quality group-design studies (randomized controlled trials and rigorous quasi-experimental designs) that meet What Works Clearinghouse (WWC) standards. Single-case design studies that meet WWC pilot standards for well-designed single-case design research are also described, but do not affect the level of evidence rating. The panel paid particular attention to a set of high-quality experimental and quasi-experimental studies that meets the WWC criteria, including both national and international studies of strategies for teaching problem solving to students in grades 4 through 8.[17] This body of research included strategies and curricular materials developed by researchers or ones commonly being used by teachers in classrooms. The panel also considered studies recommended by panel members that included students in grades 3 and 9.

Studies of problem-solving interventions in the past 20 years have yielded few causal evaluations of the effectiveness of the variety of approaches used in the field. For example, as much as the panel believes that teaching students to persist in solving challenging problems is important to solving math problems, it could not find causal research that isolated the impact of persistence. The panel also wanted to include studies of teachers using their students' culture to enhance problem-solving instruction; however, panelists could not find enough research that met WWC standards and isolated this practice. The panel was able to include suggestions for teaching the language of mathematics and for adapting problems so that contexts are more relevant to students—but these suggestions are supported by limited evidence.

The research base for this guide was identified through a comprehensive search for studies evaluating instructional practices for improving students' mathematical problem solving. An initial search for literature related to problem-solving instruction in the past 20 years yielded more than 3,700 citations; the panel recommended an additional 69 citations. Peer reviewers suggested several additional studies. Of these studies, only 38 met the causal validity standards of the WWC and were related to the panel's recommendations.[18]

The supporting research provides a strong level of evidence for two of the recommendations, a moderate level of evidence for another two of the recommendations, and a minimal level of evidence for one recommendation. Despite the varying levels of evidence, the panel believes all five recommendations are important for promoting effective problem-solving skills in students. The panel further believes that even though the level of evidence for Recommendation 1 is minimal, the practice holds promise for improving students' mathematical problem solving. Very few studies examine the effects of teacher planning on student achievement; therefore, few studies are available to support this recommendation. Nonetheless, the panel believes that the practice of intentionally preparing problem-solving lessons can lead to improvement in students' problem-solving abilities.

Table 2 shows each recommendation and the strength of the evidence that supports it as determined by the panel. Following the recommendations and suggestions for carrying out the recommendations, Appendix D presents more information on the research evidence that supports each recommendation. It also provides details on how studies were assessed as showing positive, negative, or no effects.

Table 2. Recommendations and corresponding levels of evidence

	Levels of Evidence		
Recommendation	**Strong Evidence**	**Moderate Evidence**	**Minimal Evidence**
1. Prepare problems and use them in whole-class instruction.			◆
2. Assist students in monitoring and reflecting on the problem-solving process.	◆		
3. Teach students how to use visual representations.	◆		
4. Expose students to multiple problem-solving strategies.		◆	
5. Help students recognize and articulate mathematical concepts and notation.		◆	

Prepare problems and use them in whole-class instruction.

A sustained focus on problem solving is often missing in mathematics instruction, in large part due to other curricular demands placed on teachers and students.[19] Daily math instruction is usually limited to learning and practicing new skills, leaving problem-solving time to independent seatwork or homework assignments.[20] The panel believes instruction in problem solving must be an integral part of each curricular unit, with time allocated for problem-solving activities with the whole class. In this recommendation, the panel provides guidance for thoughtful preparation of problem-solving lessons. Teachers are encouraged to use a variety of problems intentionally and to ensure that students have the language and mathematical content knowledge necessary to solve the problems.

Summary of evidence: Minimal Evidence

Few studies directly tested the suggestions of this recommendation, leading the panel to assign a level of evidence rating for this recommendation of "minimal evidence." Although the panel believes teacher planning should incorporate both routine and non-routine problems, no studies meeting WWC standards directly examined this issue.

One study found that students performed better when teacher planning considered students' mathematical content weaknesses and understanding of language and context.[21] However, this intervention included additional instructional components that may have caused the positive results. Similarly, while another study found that incorporating familiar contexts into instruction can improve problem-solving skills, the intervention included other instructional components that may have caused these positive results.[22]

On a related issue, a few well-designed studies did find that students who have practiced with word problems involving contexts (people, places, and things) they like and know do better on subsequent word-problem tests than do students who have practiced with generic contexts.[23] These achievement gains occurred when computer programs were used to personalize problem contexts for individual students or when contexts were based on the common preferences of student groups.[24]

The panel identified three suggestions for how to carry out this recommendation.

How to carry out the recommendation

1. Include both routine and non-routine problems in problem-solving activities.

Definitions of routine and non-routine problems

> **Routine** problems can be solved using methods familiar to students[25] by replicating previously learned methods in a step-by-step fashion.[26]
>
> **Non-routine** problems are problems for "which there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example."[27]
>
> Teachers must consider students' previous experience with problem solving to determine which problems will be routine or non-routine for them. A seemingly routine problem for older students or adults may present surprising challenges for younger students or those who struggle in mathematics.

Teachers should choose routine problems if their goal is to help students understand the meaning of an operation or mathematical idea. Collections of routine problems can help students understand what terms such as *multiplication* and *place value* mean, and how they are used in everyday life.[28] For example, $6 \div \frac{2}{3}$ may become more meaningful when incorporated into this word problem: "How many $\frac{2}{3}$-foot long blocks of wood can a carpenter cut from a 6-foot long board?" (See Example 1 for additional sample routine problems.)

Routine problems are not only the one- and two-step problems students have solved many times, but they can also be cognitively demanding multistep problems that require methods familiar to students. For example, see problem 3 in Example 1. The typical challenge of these problems is working through the multiple steps, rather than determining new ways to solve the problem. Thus, sixth- or seventh-grade students who have been taught the relevant geometry (e.g., types of triangles, area formulas for triangles) and basic features of coordinate graphs should be able to solve this problem by following a set of steps that may not differ significantly from what they may have already been shown or practiced. In this instance, it would be reasonable for an average student to draw a line between (0,4) and (0,10), observe that the length of this distance is 6, and then use this information in the area formula $A = \frac{1}{2} \times b \times h$. If the student substitutes appropriately for the variables in the formula, the next set is to solve for the height: $12 = \frac{1}{2} \times 6 \times h$. This step yields a height of 4, and a student could then answer the question with either (4,0) or (4,10). The routine nature of the problem solving is based on the fact that this problem may require little or no transfer from previously modeled or worked problems.

Example 1. Sample routine problems

1. Carlos has a cake recipe that calls for 2¾ cups of flour. He wants to make the recipe 3 times. How much flour does he need?

This problem is likely routine for a student who has studied and practiced multiplication with mixed numbers.

2. Solve $2y + 15 = 29$

This problem is likely routine for a student who has studied and practiced solving linear equations with one variable.

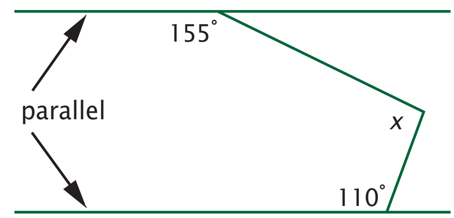
3. Two vertices of a right triangle are located at (0,4) and (0,10). The area of the triangle is 12 square units. Find a point that works as the third vertex.

This problem is likely routine for a student who has studied and practiced determining the area of triangles and graphing in coordinate planes.

When the primary goal of instruction is to develop students' ability to think strategically, teachers should choose non-routine problems that force students to apply what they have learned in a new way.[29] Example 2 provides samples of problems that are non-routine for most students. For students who have not had focused instruction on geometry problems like problem 1 in Example 2, the task presents a series of challenges. Much more time needs to be spent interpreting the problem and determining what information is relevant, as well as how it should be used. Time also needs to be spent determining or inferring if information not presented in the problem is relevant (e.g., what the measures of the supplementary angles are in the problem, how this information might be used to solve the problem). All of these features increase the cognitive demands associated with solving this problem and make it non-routine. Finally, competent students who solve this problem would also spend additional time double-checking the correctness of the solution.

Example 2. Sample non-routine problems

1. Determine angle x without measuring. Explain your reasoning.



This problem is likely non-routine for a student who has only studied simple geometry problems involving parallel lines and a transversal.

2. There are 20 people in a room. Everybody high-fives with everybody else. How many high-fives occurred?

This problem is likely non-routine for students in beginning algebra.

3. Solve for the variables a through f in the equations below, using the digits from 0 through 5. Every digit should be used only once. A variable has the same value everywhere it occurs, and no other variable will have that value.

$a + a + a = a^2$

$b + c = b$

$d \times e = d$

$a - e = b$

$b^2 = d$

$d + e = f$

The problem is likely non-routine for a student who has not solved equations by reasoning about which values can make an equation true.

4. In a leap year, what day and time are exactly in the middle of the year?

This problem is likely non-routine for a student who has not studied problems in which quantities are subdivided into unequal groups.

2. Ensure that students will understand the problem by addressing issues students might encounter with the problem's context or language.

Given the diversity of students in today's classrooms, the problems teachers select for lesson plans may include contexts or vocabulary that are unfamiliar to some.[30] These students may then have difficulty solving the problems for reasons unrelated to the students' understanding of concepts or their ability to compute answers.[31] This is a particularly significant issue for English language learners and for students with disabilities.

The goal of ensuring that students understand the language and context of problems is not to make problems less challenging. Instead, it is to allow students to focus on the mathematics in the problem, rather than on the need to learn new background knowledge or language. The overarching point is that students should understand the problem and its context before attempting to solve it.[32]

Here are some ways teachers can prepare lessons to ensure student understanding:

- **Choose problems with familiar contexts.** Students can often solve problems more successfully when they are familiar with the subject matter presented.[33]
- **Clarify unfamiliar language and contexts.** Identify the language and contexts that need to be clarified in order for students to understand the problem.[34] Teachers should think about students' experiences as they make these determinations. For example, an *island* in the kitchen, a *yacht*, the *Iditarod dog sled race*, a *Laundromat*, or sentence structures such as "*if…, then…*"[35] might need clarification, depending on students' backgrounds. Example 3 shows how one 5th-grade teacher identified vocabulary and contextual terms that needed to be clarified for her students.
- **Reword problems, drawing upon students' experiences.** Reword problems so they are familiar to students by drawing upon students' personal, familial, and community experiences.[36] Teachers can replace unfamiliar names, objects, and activities that appear in the problem with familiar ones, so as to create a problem-solving context that is more aligned with students' experiences. In this way, the problem becomes more culturally relevant and easier for students to respond to. For example, soccer may be more familiar to some students than hockey, apples more familiar than soybeans, and the guitar more familiar than the oboe. By rewording a problem to meet the needs of their students, teachers may not only increase comprehension levels, but also motivate their students to become more involved in the problem-solving activity.[37] Please note that teachers need not always reword the problems themselves. They can discuss with their students how to make the problems more familiar, interesting, and comprehensible. For instance, teachers can ask students questions such as "Can we change *Ms. Inoye* to the name of a person you know?" or "This problem says we have to measure the *shag carpet* in the living room. Can we use other words or just the word *carpet* in this problem to make it easier to understand?"

Example 3. One teacher's efforts to clarify vocabulary and context

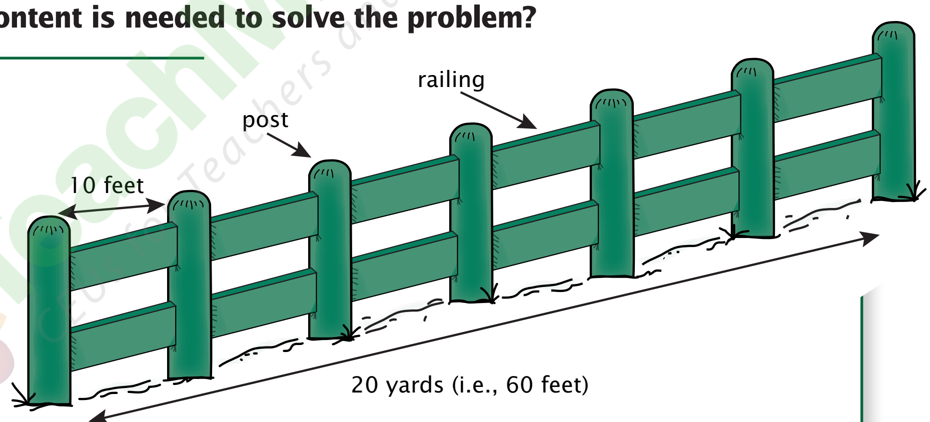
Mary, a 5th-grade teacher, identified the following vocabulary, contextual terms, and content for clarification, based on the background of her students.

Example Problem	Vocabulary	Context
In a factory, 54,650 parts were made. When they were tested, 4% were found to be defective. How many parts were working?	Students need to understand the term *defective* as being the opposite of *working* and the symbol % as *percent* to correctly solve the problem.	What is a *factory*? What does *parts* mean in this context?
At a used-car dealership, a car was priced at $7,000. The salesperson then offered a discount of $350. What percent discount, applied to the original price, gives the offered price?	Students need to know what *offered* and *original price* mean to understand the goal of the problem, and they need to know what *discount* and *percent discount* mean to understand what mathematical operators to use.	What is a *used-car dealership*?

Example 4. What mathematical content is needed to solve the problem?

Problem

Sarah Sanchez is planning to build a corral on her ranch for her two horses. She wants to build the corral in the shape of a rectangle. Here is a drawing of one side of the corral, and as you can see, this side is 20 yards wide.



It will take 480 yards of railing to build the corral based on Sarah's plan.

1. What will be the perimeter of the corral?
2. What will be the area of the corral?
3. Can you show a way that Sarah can use the same amount of railing and build a corral with a bigger area for her horses?

Mathematical content needed for solution

- Addition, subtraction, multiplication and division.
- Opposite sides of rectangles are equal.
- The perimeter of a shape can stay the same, but its area can change.

3. Consider students' knowledge of mathematical content when planning lessons.

As teachers plan problem-solving instruction, they should identify the mathematical content needed to solve the included problems. It is important to remember that problems that align with the current unit often draw on skills taught in prior units or grade levels. In the problems listed in Example 3, students need to understand (1) what *percent* means (i.e., that 4% is 4 out of 100 or $4/100$ or 0.04), and (2) how to calculate the value of a percent of a quantity or percent of a change. Struggling students are likely to benefit from a quick review of the relevant skills needed to understand and solve the problem, especially if the mathematical content has not been discussed recently or if a non-routine problem is presented.[38] A brief review of skills learned earlier also helps students see how this knowledge applies to challenging problems.[39]

For example, teachers may need to review the difference between fractions and ratios before students are asked to solve ratio and proportion problems. Similarly, before solving the fourth-grade problem shown in Example 4, students should know and be able to apply several facts about area and perimeter.

Mathematical language in problems may also need to be reviewed with students. For example, before presenting students with a problem such as the one in Example 5, a teacher might need to clarify the terms in the problem.

Example 5. Mathematical language to review with students

Problem

Two vertices of a triangle are located at (0,4) and (0,10). The area of the triangle is 12 square units. What are all possible positions for the third vertex?

Mathematical language that needs to be reviewed

- *vertices*
- *area square units*
- *triangle*
- *vertex*

Potential roadblocks and solutions

Roadblock 1.1. *Teachers are having trouble finding problems for the problem-solving activities.*

Suggested Approach. Textbooks usually include both routine and non-routine problems, but teachers often have a hard time finding non-routine problems that fit their lesson's goals. In addition to the class text, teachers may need to use ancillary materials, such as books on problem solving and handouts from professional-development activities. Teachers also can ask colleagues for additional problem-solving activities or work on teams with other teachers or with instructional leaders using lesson study to prepare materials for problem-solving instruction. Teachers also can search the Internet for examples. Helpful online resources include *Illuminations* from the National Council of Teachers of Mathematics, problems of the week from the Math Forum at Drexel University, and practice problems from high-quality standardized tests such as the state assessments, the Trends in International Mathematics and Science Study (TIMSS), the Programme for International Student Assessment (PISA), and the Scholastic Assessment Test (SAT).[40]

Roadblock 1.2. *Teachers have no time to add problem-solving activities to their mathematics instruction.*

Suggested Approach. The panel believes that including problem-solving activities throughout each unit is essential. To make time during instruction, teachers should consider balancing the number of problems students are required to solve during seatwork activities with worked examples students can simply study. Worked examples could benefit student learning and decrease the time necessary to learn a new skill.[41] For more information on how to use worked examples as a part of problem-solving instruction, see Recommendations 2 and 4 of this practice guide or Recommendation 2 of the *Organizing Instruction and Study to Improve Student Learning* practice guide.[42]

Overview of Recommendation 2 in the *Organizing Instruction and Study to Improve Student Learning* practice guide[43]

1. Teachers should give students assignments that provide already worked solutions for students to study, interleaved with problems for them to solve on their own.
2. As students develop greater problem-solving abilities, teachers can reduce the number of worked problems they provide and increase the number of problems that students should solve independently.

Roadblock 1.3. *Teachers are not sure which words to teach when teaching problem solving.*

Suggested Approach. The panel believes academic language, including the language used in mathematics, should be taught explicitly so that all students understand what is being asked in a problem and how the problem should be solved. Identifying the language used in a problem-solving task can guide lesson planning. Based on the scope and sequence of the curricular material, math coaches or specialists can provide a list of academic words and phrases (e.g., *addition, not greater than*)[44] that are essential for teaching a given unit. The list can also focus on the language that will be necessary for students to know as they progress to the next grade level. Teachers can work with colleagues to solve problems and identify words students need to understand to solve the problem. They also can look for important academic terms and vocabulary in the class textbook or the mathematics standards for the state.

Assist students in monitoring and reflecting on the problem-solving process.

Students learn mathematics and solve problems better when they monitor their thinking and problem-solving steps as they solve problems.[45] *Monitoring and reflecting during problem solving helps students think about what they are doing and why they are doing it, evaluate the steps they are taking to solve the problem, and connect new concepts to what they already know. The more students reflect on their problem-solving processes, the better their mathematical reasoning—and their ability to apply this reasoning to new situations—will be.*[46]

In this recommendation, the panel suggests that teachers help students learn to monitor and reflect on their thought process when they solve math problems. While the ultimate goal is for students to monitor and reflect on their own while solving a problem, teachers may need to support students when a new activity or concept is introduced. For instance, a teacher may provide prompts and use them to model monitoring and reflecting as the teacher solves a problem aloud. In addition, a teacher can use what students say as a basis for helping the students improve their monitoring and reflecting. Teachers can use students' ideas to help students understand the problem-solving process.

Summary of evidence: Strong Evidence

Several studies with diverse student samples directly tested this recommendation and consistently found positive effects. As a result, the panel determined there was strong evidence to support this recommendation.[47]

The relevant studies examined students' mathematics achievement in different content areas, including numbers and operations, data analysis and probability, algebra, and geometry. Two studies found that providing students with a task list that identified specific steps to solving problems resulted in better student achievement.[48] Two additional studies found that a self-questioning checklist improved achievement,[49] and in one study, this effect persisted for at least four months after instruction ended;[50] however, both studies included additional instructional components (visual aids and multiple-strategy instruction) that may have produced the positive results. Similarly, five studies found that student performance improved when teachers modeled a self-questioning process and then asked students to practice it.[51]

The panel identified three suggestions for how to carry out this recommendation.

How to carry out the recommendation

1. Provide students with a list of prompts to help them monitor and reflect during the problem-solving process.

The prompts that teachers provide can either be questions that students should ask and answer as they solve problems (see Example 6) or task lists that help students complete steps in the problem-solving process (see Example 7).[52] The questions teachers provide should require students to think through the problem-solving process, similar to the way in which task lists guide students through the process. Select a reasonable number of prompts, rather than an exhaustive list, as too many prompts may slow down the problem-solving process or be ignored. Ensure that the prompts help students evaluate their work at each stage of the problem-solving process, from initially reading and understanding the problem, to determining a way to solve the problem, and then to evaluating the appropriateness of the solution given the facts in the problem.[53]

Encourage students to explain and justify their response to each prompt, either orally[54] or in writing.[55] Students can use the prompts when working independently, in small groups,[56] or even when solving problems at a computer.[57] When working in small groups, students can take turns asking and answering questions or reading each action aloud and responding to it. As they share in small groups, students serve as models for others in their group, allowing all the students to learn from one another. Teachers may wish to post prompts on the board, include them on worksheets,[58] or list them on index cards for students.[59]

When students first use the prompts, they may need help. Teachers can participate in the questioning or refer to tasks in the task list when students work in small groups or during whole-group discussions. If, for example, a student solves a problem incorrectly, ask him what questions he should have asked himself to help him reason out loud, rather than providing him with the correct answer.[60] Alternatively, provide the correct answer, but ask the student to explain why it is right and why his original answer is not.[61] As students become more comfortable with their reasoning abilities and take greater responsibility for monitoring and reflecting during problem solving, teachers can gradually withdraw the amount of support they provide.[62]

Example 6. Sample question list

- What is the story in this problem about?
- What is the problem asking?
- What do I know about the problem so far? What information is given to me? How can this help me?
- Which information in the problem is relevant?
- In what way is this problem similar to problems I have previously solved?
- What are the various ways I might approach the problem?
- Is my approach working? If I am stuck, is there another way I can think about solving this problem?
- Does the solution make sense? How can I verify the solution?
- Why did these steps work or not work?
- What would I do differently next time?

Note: These are examples of the kinds of questions that a teacher can use as prompts to help students monitor and reflect during the problem-solving process. Select those that are applicable for your students, or formulate new questions to help guide your students.

Example 7. Sample task list[63]

- Identify the givens and goals of the problem.
- Identify the problem type.
- Recall similar problems to help solve the current problem.
- Use a visual to represent and solve the problem.
- Solve the problem.
- Check the solution.

2. Model how to monitor and reflect on the problem-solving process.

Model how to monitor and reflect while solving a problem using the prompts given to students.[64] This can be done when introducing a problem-solving activity or a new concept to the whole class or as students work independently or in small groups.[65] Say aloud not only the response to each prompt, but also the reasons why each step was taken. Alternatively, say which step was taken, but ask students to explain why this would work. Make sure to use a prompt at each stage in the problem-solving process, for example, when first reading the problem, when attempting a strategy to solve the problem, and after solving the problem.

Example 8 describes one teacher's experience with modeling how to monitor and reflect using questions. It illustrates the importance of persistence if the student fails to understand the problem or the appropriate method to employ for solving it.

Example 8. One way to model monitoring and reflecting using questions

Problem

Last year was unusually dry in Colorado. Denver usually gets 60 inches of snow per year. Vail, which is up in the mountains, usually gets 350 inches of snow. Both places had 10 inches of snow less than the year before. Kara and Ramon live in Colorado and heard the weather report. Kara thinks the decline for Denver and Vail is the same. Ramon thinks that when you compare the two cities, the decline is different. Explain how both people are correct.

Solution

TEACHER: First, I ask myself, **"What is this story about, and what do I need to find out?"** I see that the problem has given me the usual amount of snowfall and the change in snowfall for each place, and that it talks about a decline in both cities. I know what *decline* means: "a change that makes something less." Now I wonder how the decline in snowfall for Denver and Vail can be the same for Kara and different for Ramon. I know that a decline of 10 inches in both cities is the same, so I guess that's what makes Kara correct. How is Ramon thinking about the problem?

I ask myself, **"Have I ever seen a problem like this before?"** As I think back to the assignments we had last week, I remember seeing a problem that asked us to calculate the discount on a $20 item that was on sale for $15. I remember we had to determine the percent change. This could be a similar kind of problem. This might be the way Ramon is thinking about the problem.

Before I go on, I ask myself, **"What steps should I take to solve this problem?"** It looks like I need to divide the change amount by the original amount to find the percent change in snowfall for both Denver and Vail.

Denver: $10 \div 60 = 0.166$ or 16.67% or 17% when we round it to the nearest whole number

Vail: $10 \div 350 = 0.029$ or 2.9% or 3% when we round it to the nearest whole number

So the percent decrease in snow for Denver was much greater (17%) than for Vail (3%). Now I see what Ramon is saying! It's different because the percent decrease for Vail is much smaller than it is for Denver.

Finally, I ask myself, **"Does this answer make sense when I reread the problem?"** Kara's answer makes sense because both cities did have a decline of 10 inches of snow. Ramon is also right because the percent decrease for Vail is much smaller than it is for Denver. Now, both of their answers make sense to me.

3. Use student thinking about a problem to develop students' ability to monitor and reflect.

The panel believes that, by building on students' ideas, teachers can help students clarify and refine the way they monitor and reflect as they solve a problem. Teachers can help students verbalize other ways to think about the problem. The teacher-student dialogue can include guided questioning to help students clarify and refine their thinking and to help them establish a method for monitoring and reflecting that makes sense to them (see Example 9). This is helpful for students who dislike working with teacher-provided prompts or who are having difficulty understanding and using these prompts.

Example 9. Using student ideas to clarify and refine the monitoring and reflecting process

Problem

Find a set of five different numbers whose average is 15.

Solution

TEACHER: Jennie, what did you try?

STUDENT: I'm guessing and checking. I tried 6, 12, 16, 20, 25 and they didn't work. The average is like 17.8 or something decimal like that.

TEACHER: That's pretty close to 15, though. Why'd you try those numbers?

STUDENT: What do you mean?

TEACHER: I mean, where was the target, 15, in your planning? It seems like it was in your thinking somewhere. If I were choosing five numbers, I might go with 16, 17, 20, 25, 28.

STUDENT: But they wouldn't work—you can tell right away.

TEACHER: How?

STUDENT: Because they are all bigger than 15.

TEACHER: So?

STUDENT: Well, then the average is going to be bigger than 15.

TEACHER: Okay. That's what I meant when I asked "Where was 15 in your planning?" You knew they couldn't all be bigger than 15. Or they couldn't all be smaller either?

STUDENT: Right.

TEACHER: Okay, so keep the target, 15, in your planning. How do you think five numbers whose average is 15 relate to the number 15?

STUDENT: Well, some have to be bigger and some smaller. I guess that is why I tried the five numbers I did.

TEACHER: That's what I guess, too. So, the next step is to think about how much bigger some have to be, and how much smaller the others have to be. Okay?

STUDENT: Yeah.

TEACHER: So, use that thinking to come up with five numbers that work.

Potential roadblocks and solutions

Roadblock 2.1. *Students don't want to monitor and reflect; they just want to solve the problem.*

Suggested Approach. The panel believes that students need to develop the habit of monitoring and reflecting throughout the problem-solving process, from setting up the problem to evaluating whether their solution is accurate. Ideally, whenever students solve problems, they should practice monitoring and reflecting on their problem-solving process. Acknowledge that simply solving a problem may seem easier, but encourage students to incorporate monitoring and reflecting into their process every time they solve a problem. Inform students that doing so will help them understand and solve problems better, as well as help them convey their strategies to classmates.[66] Explain that expert problem solvers learn from unsuccessful explorations and conjectures by reflecting on why they were unsuccessful.

Roadblock 2.2. *Teachers are unclear on how to think aloud while solving a non-routine problem.*

Suggested Approach. Prepare ahead of time using the list of prompts given to students. Outline responses to the prompts in advance of the lesson. It might also help to anticipate how students would think about the prompts as they solved the problem. A colleague or math coach can help teachers think through the prompts and the problem-solving process if they get stuck.

Roadblock 2.3. *Students take too much time to monitor and reflect on the problem-solving process.*

Suggested Approach. It is likely that when students initially learn the tasks of monitoring and reflecting, they will be slow in using the prompts. However, after a bit of practice, they are likely to become more efficient at using the prompts.

Roadblock 2.4. *When students reflect on problems they have already solved, they resort to using methods from those problems rather than adapting their efforts to the new problem before them.*

Suggested Approach. While students should consider whether they have seen a similar problem before, sometimes they overdo it and simply solve the problem using similar methods, rather than using methods that will work for the problem they are solving. To help students overcome this, try asking them to explain why the solution method worked for the previous problem and what components of it may or may not be useful for the new problem.

Teach students how to use visual representations.

A major task for any student engaged in problem solving is to translate the quantitative information in a problem into a symbolic equation—an arithmetic/algebraic statement—necessary for solving the problem. Visual representations help students solve problems by linking the relationships between quantities in the problem with the mathematical operations needed to solve the problem. Students who learn to visually represent the mathematical information in problems prior to writing an equation are more effective at problem solving.[67]

Visual representations include tables, graphs, number lines, and diagrams such as strip diagrams, percent bars, and schematic diagrams. Example 10 provides a brief explanation of how a few types of visual representations can be used to solve problems.[68] *In the panel's opinion, teachers should consistently teach students to use a few types of visual representations rather than overwhelming them with many examples. In this recommendation, the panel offers suggestions for selecting appropriate visual representations to teach and methods for teaching students how to represent the problem using a visual representation.*

Definitions of strip diagrams, percent bars, and schematic diagrams

Strip diagrams use rectangles to represent quantities presented in the problem.

Percent bars are strip diagrams in which each rectangle represents a part of 100 in the problem.

Schematic diagrams demonstrate the relative sizes and relationships between quantities in the problem.

Example 10. Sample table, strip diagram, percent bar, and schematic diagram

Problem

Cheese costs \$2.39 per pound. Find the cost of 0.75 pounds of cheese.[69]

Sample table

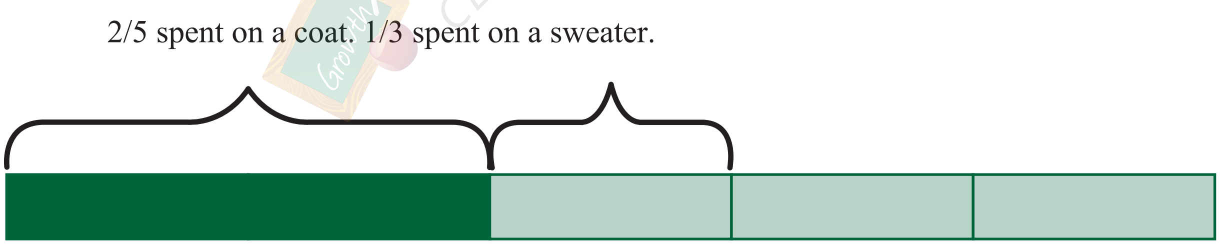
	Cost of Cheese		Pounds of Cheese
(a)	2.39		1
	?		0.75
(b)	2.39	← 2.39 ←	1
	x	← 2.39 ←	0.75

This table depicts the relationship between the weight of cheese and its cost. Every pound of cheese will cost \$2.39, and this relationship can be used to determine the cost of 0.75 pounds of cheese by using the rule "times 2.39," which can be stated in an equation as $x = 0.75 \times 2.39$.

Problem

Eva spent ⅖ of the money she had on a coat and then spent ⅓ of what was left on a sweater. She had \$150 remaining. How much did she start with?

Sample strip diagram



This strip diagram depicts the money Eva spent on a coat and a sweater. It shows how the amount of money she originally had is divided into 5 equal parts and that 2 of the 5 parts are unspent. The problem states that the unspent amount equals \$150. Several strategies can then be employed to make use of this information in an equation, such as $⅖ \times x = 150$, to determine the original amount.

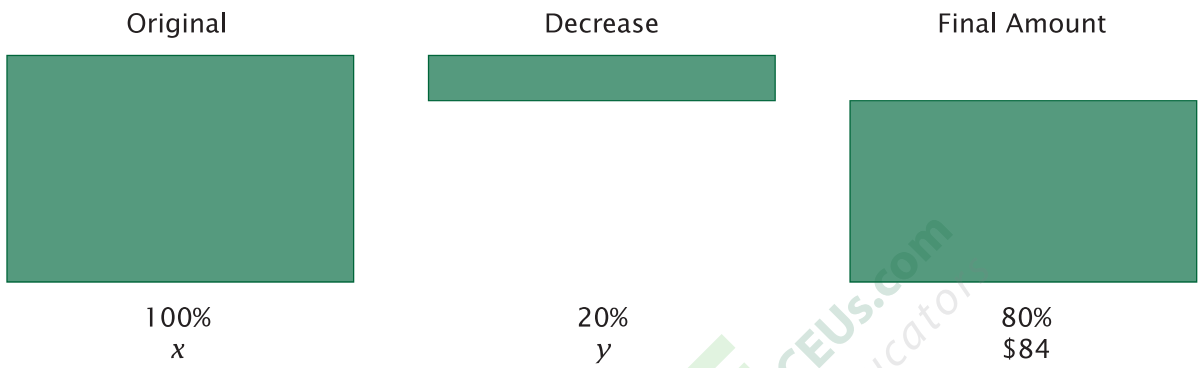
(continued)

Example 10. Sample table, strip diagram, percent bar, and schematic diagram *(continued)*

Problem

During a sale, prices were marked down by 20%. The sale price of an item was \$84. What was the original price of the item before the discount?[70]

Sample percent bar



These percent bars depict the relative values of the original, decrease, and final amounts as 100:20:80, which can be reduced to 5:1:4. The relationship between the original and final amount (5:4) can be used in an algebraic equation, such as $x/84 = 5/4$, to determine the original amount when the final amount is \$84.

Problem

John recently participated in a 5-mile run. He usually runs 2 miles in 30 minutes. Because of an ankle injury, John had to take a 5-minute break after every mile. At each break he drank 4 ounces of water. How much time did it take him to complete the 5-mile run?

Sample schematic diagram

This schematic diagram depicts the amount of time John needed to run 5 miles when each mile took him 15 minutes to run and he took a 5-minute break after every mile. The total time (x) it took him to complete the run is equal to the total number of minutes in this diagram, or $x = (5 \times 15) + (4 \times 5)$.

Summary of evidence: Strong Evidence

The panel determined there is strong evidence supporting this recommendation because six studies with middle school student samples consistently found that using visual representations improved achievement.[71] Both general education students and students with learning disabilities performed better when taught to use visual representations[72] such as identifying and mapping relevant information onto schematic diagrams.[73]

In four of the six studies, students were taught to differentiate between types of math problems and then to implement an appropriate diagram for the relevant type.[74] An additional study involving an alternative problem-solving approach integrated with visual representations also resulted in higher achievement.[75] Finally, one study showed that if teachers help students design, develop, and improve their own visual representations, student achievement improves more than if students simply use teacher- or textbook-developed visuals.[76]

The panel identified three suggestions for how to carry out this recommendation.

How to carry out the recommendation

1. Select visual representations that are appropriate for students and the problems they are solving.

Sometimes curricular materials suggest using more than one visual representation for a particular type of problem. Teachers should not feel obligated to use all of these; instead, teachers should select the visual representation that will work best for students and should use it consistently for similar problems.[77]

For example, suppose a teacher introduced a ratio or proportion problem using a diagram that students found helpful in arriving at the equation needed to solve the problem. The teacher should continue using this same diagram when students work on additional ratio or proportion problems. Remember, students may need time to practice using visual representations and may struggle before achieving success with them.[78] If, after a reasonable amount of time and further instruction, the representation still is not working for individual students or the whole class, consider teaching another type of visual representation to the students in the future. Teachers can also consult a mathematics coach, other math teachers, or practitioner publications to identify more appropriate visual representations.

Also keep in mind that certain visual representations are better suited for certain types of problems.[79] For instance, schematic diagrams work well with ratio and proportion problems, percent bars are appropriate for percent problems, and strip diagrams are suited for comparison and fraction problems.

2. Use think-alouds and discussions to teach students how to represent problems visually.

When teaching a new visual representation or type of problem, demonstrate how to represent the problem using the representation. Teachers should think aloud about the decisions they make as they connect the problem to the representation.[80] Thinking aloud is more than just the teacher telling students what he or she is doing. It also involves the teacher expressing his or her thoughts as he or she approaches the problem, including what decisions he or she is making and why he or she is making each decision (see Example 11).

Teachers should explain how they identified the type of problem—such as proportion, ratio, or percent—based on mathematical ideas in the problem and why they think a certain visual representation is most appropriate. For example, proportion problems describe equality between two ratios or rates that allows students to think about how both are the same. Teachers should be careful not to focus on surface features such as story context.[81] In Example 12, the story contexts are different, but the problem type is the same. Students who cannot articulate the type of problem may struggle to solve it, even if they have the basic math skills.[82]

Demonstrate to students how to represent the information in a problem visually.[83] Teach students to identify what information is relevant or critical to solving the problem. Often, problems include information that is irrelevant or unnecessary. For instance, in Example 11, students need to determine how many red roses are in Monica's bouquet. The number of pink roses in Bianca's bouquet is irrelevant; students need only focus on Monica's bouquet.

Example 11. One way of thinking aloud[84]

Problem

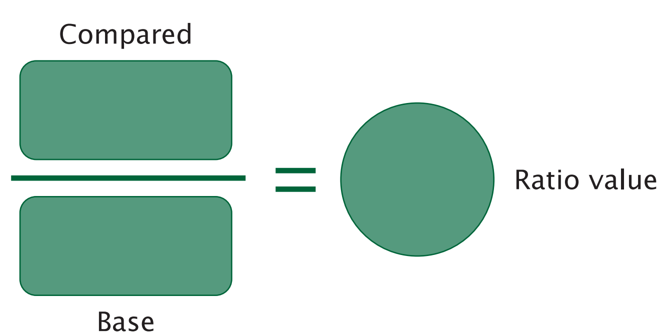
Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

Solution

TEACHER: I know this is a ratio problem because two quantities are being compared: the number of red roses and the number of yellow roses. I also know the ratio of the two quantities. There are 3 red roses for every 5 yellow roses. This tells me I can find more of each kind of rose by multiplying.

I reread the problem and determine that I need to solve the question posed in the last sentence: "How many red roses are in Monica's bouquet?" Because the question is about Monica, perhaps I don't need the information about Bianca. The third sentence says there are two dozen red and yellow roses. I know that makes 24 red and yellow roses, but I still don't know how many red roses there are. I know there are 3 red roses for every 5 yellow roses. I think I need to figure out how many red roses there are in the 24 red and yellow roses.

Let me reread the problem… That's correct. I need to find out how many red roses there are in the bouquet of 24 red and yellow roses. The next part of the problem talks about the ratio of red roses to red and yellow roses. I can draw a diagram that helps me understand the problem. I've done this before with ratio problems. These kinds of diagrams show the relationship between the two quantities in the ratio.

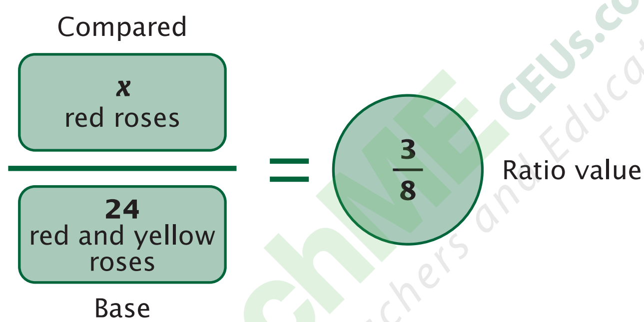


(continued)

Example 11. One way of thinking aloud *(continued)*

TEACHER: I write the quantities and units from the problem and an *x* for what must be solved in the diagram. First, I am going to write the ratio of red roses to yellow roses here in the circle. This is a part-to-whole comparison—but how can I find the whole in the part-to-whole ratio when we only know the part-to-part ratio (the number of red roses to the number of yellow roses)?

I have to figure out what the ratio is of red roses to red and yellow roses when the problem only tells about the ratio of red roses to yellow roses, which is 3:5. So if there are 3 red roses for every 5 yellow roses, then the total number of units for red and yellow roses is 8. For every 3 units of red roses, there are 8 units of red and yellow roses, which gives me the ratio 3:8. I will write that in the diagram as the ratio value of red roses to red and yellow roses. There are two dozen red and yellow roses, and that equals 24 red and yellow roses, which is the base quantity. I need to find out how many red roses (x) there are in 24 red and yellow roses.



I can now translate the information in this diagram to an equation like this:

$$\frac{x \text{ red roses}}{24 \text{ red-and-yellow roses}} = \frac{3}{8}$$

Then, I need to solve for x.

$$\frac{x}{24} = \frac{3}{8}$$

$$24\left(\frac{x}{24}\right) = 24\left(\frac{3}{8}\right)$$

$$x = \frac{72}{8}$$

$$x = 9$$

Example 12. Variations in story contexts for a proportion problem

Problem

Solve $\frac{2}{10} = \frac{x}{30}$

Context 1

Sara draws 2 trees for every 10 animals. How many trees will she need to draw if she has 30 animals?

Context 2

Sarah creates a tiled wall using 2 black tiles for every 10 white tiles. If she has 30 white tiles, how many black tiles will she need?

Promote discussions by asking students guiding questions as they practice representing problems visually.[85] For example, teachers can ask the following questions:

- What kind of problem is this? How do you know?
- What is the relevant information in this problem? Why is this information relevant?
- Which visual representation did you use when you solved this type of problem last time?
- What would you do next? Why?

Encourage students to discuss similarities and differences among the various visuals they have learned or used. When students use their own visual representations to solve a problem correctly, teachers can emphasize noteworthy aspects of the representations and ask the students to share their visual representations with the class. They also may ask the students to explain how and why they used a particular representation to solve a problem. By sharing their work, students are modeling how to use visual representations for other students, allowing them to learn from one another.

3. Show students how to convert the visually represented information into mathematical notation.

After representing the relevant information in a problem visually, demonstrate how each part of the visual representation can be translated into mathematical notation.[86] Students must see how each quantity and relationship in the visual representation corresponds to quantities and relationships in the equation.

Sometimes, the translation from representation to equation is as simple as rewriting the quantities and relationships without the boxes, circles, or arrows in the visual representation (see Example 11). Other times, correspondence between the visual representation and the equation is not as simple, and teachers must illustrate the connections explicitly. For example, a teacher using the table in Example 10 should demonstrate how to represent the cost of 0.75 pounds of cheese as *x* and the rule "times 2.39" with the correct notation in an equation.

Potential roadblocks and solutions

Roadblock 3.1. *Students do not capture the relevant details in the problem or include unnecessary details when representing a problem visually.*

Suggested Approach. Often, when representing a problem visually, students do not capture the relevant details and relationships in the problem, or include unnecessary details, such as the dots on a ladybug or a picket fence around a house. Consequently, such representations do not correctly depict or help identify the mathematical nature of the problem. Teachers can help students improve their representations by building upon students' thinking. They can ask guiding questions that will help students clarify and refine their representations. Once a student has revised his or her representation, teachers can ask him or her to explain what was missing from the representation and why the representation did not work initially. Teachers should be sure to point out specific aspects of the representation that the student did correctly. This will encourage students to keep trying.

If necessary, teachers also can demonstrate how to alter the representation to represent and solve the problem correctly, using students' representations as a springboard for refinement.[87] Teachers can show students how their diagrams can be modified to represent the relevant information without the unnecessary details. It is important to help students improve their representations, because students who represent irrelevant information could be less effective problem solvers than those who draw diagrams with relevant details from the problem.[88] Teachers also can help by explaining what the difference is between relevant and irrelevant details and how a visual representation can capture relevant details and relationships. They also should emphasize that a diagram's goal is to illustrate the relationships that are important for solving the problem.

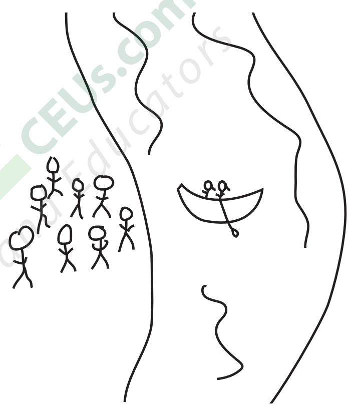
Consider the river-crossing problem detailed in Example 13. The first representation is a simple narrative description of the story that depicts the boat and river, and the adults and children waiting to cross the river. The schematic diagram, on the other hand, outlines the sequence of trips and boat occupants that are needed to take 4 adults and 2 children across the river in a small boat. The schematic diagram better represents the relevant information in the problem and is helpful in arriving at the symbolic equation.

Example 13. Diagrams with relevant and irrelevant details[89]

Problem

There are 4 adults and 2 children who need to cross the river. A small boat is available that can hold either 1 adult or 1 or 2 small children. Everyone can row the boat. How many one-way trips does it take for all of them to cross the river?

Using a representation without relevant details and with irrelevant details to represent the problem:



Using a schematic diagram with relevant details and without irrelevant details to represent the problem:

Start

Trip	Left Bank Adults	Left Bank Kids	River boat trip	Right Bank Adults	Right Bank Kids
1	4 Adults	0 kids	→ 2 kids →	0 Adults	2 kids
2	4A	1K	← 1 kid ←	0A	1 K
3	3 A	1k	→ 1 Adult →	1	1
4	3A	2k	← 1Kid ←	1	0
5	3	0	→ 2 kid →	1	2
6	3	1	← 1K ←	1	1
7	2	1	→ 1A →	2	1
8	2	2	← 1k ←	2	0
9	2	0	→ 2k →	2	2
10	2	1	← 1k ←	2	1
11	1	1	→ 1 A →	3	1
12	1	2	← 1k ←	3	0
13	1	0	→ 2K →	3	2
14	1	1	← 1k ←	3	1
15	0	1	→ 1A →	4	1
16	0	2	← 1k ←	4	0
17	0	0	→ 2K →	4	2

Finish

Roadblock 3.2. *The class text does not use visual representations.*

Suggested Approach. Teachers can ask colleagues or math coaches for relevant visual representations, or they can develop some on their own. Teachers also can look through professional-development materials they may have collected or search the Internet for more examples. Using an overhead projector, a document reader, or an interactive white-board, teachers can incorporate these visuals into their lessons.

Expose students to multiple problem-solving strategies.

Problem solvers who know how to use multiple strategies to solve problems may be more successful.[90] *When regularly exposed to problems that require different strategies, students learn different ways to solve problems. As a result, students become more efficient in selecting appropriate ways to solve problems*[91] *and can approach and solve math problems with greater ease and flexibility.*[92]

In this recommendation, the panel suggests ways to teach students that problems can be solved in more than one way and that they should learn to choose between strategies based upon their ease and efficiency. The panel recommends that teachers instruct students in a variety of strategies for solving problems and provide opportunities for students to use, share, and compare the strategies. Teachers should consider emphasizing the clarity and efficiency of different strategies when they are compared as part of a classroom discussion.

Summary of evidence: Moderate Evidence

Eight studies found positive effects of teaching and encouraging multiple problem-solving strategies, although in some of these studies the effects were not discernible across all types of outcomes. Three additional studies involving students with limited or no knowledge of algebraic methods found negative effects for some algebra outcomes—two of these studies also found positive effects for some outcomes. Consequently, the panel determined that there is moderate evidence to support this recommendation.

Six of the seven studies that included procedural flexibility outcomes found that exposing students to multiple problem-solving strategies improved students' procedural flexibility—their ability to solve problems in different ways using appropriate strategies.[93] However, the estimated effects of teaching multiple strategies on students' ability to solve problems *correctly* (procedural knowledge) and awareness of mathematical concepts (conceptual knowledge) were inconsistent.[94]

Three studies found that when students were instructed in using multiple strategies to solve the same problem, procedural knowledge improved; however, all of these studies included additional instructional components (checklists and visual aids) that may have produced the positive results.[95] Another study with an eight-minute strategy demonstration found no discernible effects.[96] Providing students with worked examples explicitly comparing multiple-solution strategies had positive effects on students' procedural flexibility in three of the four studies that examined this intervention; however, the three studies found inconsistent effects on students' procedural knowledge and conceptual knowledge.[97] The fourth study providing students with worked examples found that the effects varied by baseline skills—the intervention had a negative effect on procedural knowledge, conceptual knowledge, and procedural flexibility for students who did not attempt algebra reasoning on a pretest, but no discernible effect for students who had attempted algebraic reasoning on the pretest (correctly or incorrectly).[98] Finally, when students attempted to solve problems using multiple strategies and then shared and compared their strategies, their ability to solve problems did improve.[99] Two additional studies involving students with no or minimal algebra knowledge found that asking students to re-solve an algebra problem using a different method had negative effects on procedural knowledge measures and positive effects on procedural flexibility.[100]

The panel identified three suggestions for how to carry out this recommendation.

How to carry out the recommendation

1. Provide instruction in multiple strategies.

Teach students multiple strategies for solving problems.[101] These can be problem-specific[102] or general strategies for use with more than one type of problem.[103] For instance, in Example 14, a teacher shows his or her students two ways to solve the same problem.

Example 14. Two ways to solve the same problem

Problem

Ramona's furniture store has a choice of 3-legged stools and 4-legged stools. There are five more 3-legged stools than 4-legged stools. When you count the legs of the stools, there are exactly 29 legs. How many 3-legged and 4-legged stools are there in the store?

Solution 1: Guess and check

4×4 legs = 16 legs	9×3 legs = 27 legs	total = 43 legs
3×4 legs = 12 legs	8×3 legs = 24 legs	total = 36 legs
2×4 legs = 8 legs	7×3 legs = 21 legs	total = 29 legs

TEACHER: This works; the total equals 29, and with two 4-legged stools and seven 3-legged stools, there are five more 3-legged stools than 4-legged stools.

(continued)

Example 14. Two ways to solve the same problem *(continued)*

Solution 2

TEACHER: Let's see if we can solve this problem logically. The problem says that there are five more 3-legged stools than 4-legged stools. It also says that there are 29 legs altogether. If there are five more 3-legged stools, there has to be at least one 4-legged stool in the first place. Let's see what that looks like.

stools		
total legs	4 × 1 = 4 +	3 × 6 = 18
	4 + 18 = 22	

TEACHER: We can add a stool to each group, and there will still be a difference of five stools.

stools		
total legs	4 × 2 = 8 +	3 × 7 = 21
	8 + 21 = 29	

TEACHER: I think this works. We have a total of 29 legs, and there are still five more 3-legged stools than 4-legged stools. We solved this by thinking about it logically. We knew there was at least one 4-legged stool, and there were six 3-legged stools. Then we added to both sides so we always had a difference of five stools.

When teaching multiple strategies, periodically employ unsuccessful strategies and demonstrate changing to an alternate strategy to show students that problems are not always solved easily the first time and that sometimes problem solvers need to try more than one strategy to solve a problem. This will help students develop the persistence the panel believes is necessary to complete challenging and non-routine problems.

2. Provide opportunities for students to compare multiple strategies in worked examples.

Worked examples allow for quick, efficient comparisons between strategies.[104] Successful students should be able to compare the similarities and differences among multiple strategies.[105] They may reap more benefits from comparing multiple strategies in worked examples when they work with a partner instead of alone[106] and when they can actively participate in the learning process.[107] Teachers should provide opportunities for students to work together and should use worked examples to facilitate the comparison of strategies.

Teachers can use worked examples to facilitate comparison of strategies with interesting contrasts and not just minor differences. An added benefit of comparing strategies is that certain examples allow for concepts to be highlighted. The strategies used in Example 15 allow for a discussion of treating $(y + 1)$ as a composite variable.

Teachers should present worked examples side-by-side on the same page, rather than on two separate pages, to facilitate more effective comparisons (see Example 15).[108] Teachers also should include specific questions to facilitate student discussions.[109] For example, teachers can ask these questions:

- How are the strategies similar? How are they different?
- Which method would you use to solve the problem? Why would you choose this approach?
- The problem was solved differently, but the answer is the same. How is that possible?

Ask students to respond to the questions first verbally and then in writing.[110] See Example 15 for an example of how these questions can be tailored to a specific problem.

Example 15. A comparison of strategies[111]

Mandy's solution		Erica's solution	
$5(y + 1) = 3(y + 1) + 8$		$5(y + 1) = 3(y + 1) + 8$	
$5y + 5 = 3y + 3 + 8$	Distribute	$2(y + 1) = 8$	Subtract on both
$5y + 5 = 3y + 11$	Combine	$y + 1 = 4$	Divide on both
$2y + 5 = 11$	Subtract on both	$y = 3$	Subtract on both
$2y = 6$	Subtract on both		
$y = 3$	Divide on both		

TEACHER: Mandy and Erica solved the problem differently, but they got the same answer. Why? Would you choose to use Mandy's way or Erica's way? Why?

Worked examples, of course, should be used alongside opportunities for students to solve problems on their own. For instance, teachers can provide worked examples for students to study with every couple of practice problems. Students who receive worked examples early on in a lesson may experience better learning outcomes with less effort than students who only receive problems to solve.[112]

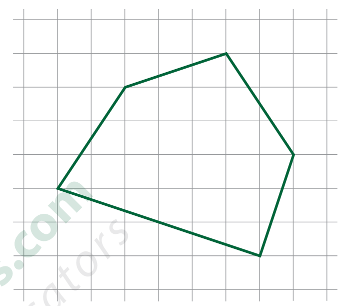
3. Ask students to generate and share multiple strategies for solving a problem.

Encourage students to generate multiple strategies as they work independently[113] or in small groups.[114] In Example 16, students share their strategies in a small group. Provide opportunities for students to share their strategies with the class. When students see the various methods employed by others, they learn to approach and solve problems in different ways.

Example 16. How students solved a problem during a small-group activity

Problem[115]

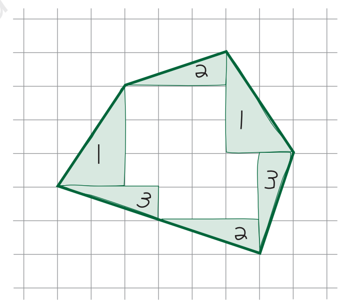
Find the area of this pentagon.



Solution strategies

Ali and Maria each worked on this problem individually. After 20 minutes in a small-group activity, they talked to each other about how they approached the problem.

ALI: The pentagon is slanted, so first I looked for figures for which I knew how to compute the area. Look what I found: six right triangles inside; and they get rid of the slanted parts, so what's left are rectangles.

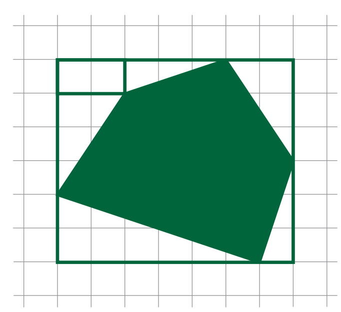


Then, I noticed that the right triangles are really three pairs of congruent right triangles. So together, the ones marked *1* have an area of $2 \times 3 = 6$ square units. The ones marked *2* combine for an area of $3 \times 1 = 3$ square units. The ones marked *3* also combine for an area of 3 square units.

What's left inside is a 2-by-3 rectangle, with an area of 6 square units; a 1-by-4 rectangle, with an area of 4 square units; and a 1-by-3 rectangle, with an area of 3 square units.

So, the area of the pentagon is $6 + 3 + 3 + 6 + 4 + 3 = 25$ square units.

MARIA: You looked inside the pentagon, but I looked outside to deal with the slanted parts. I saw that I could put the pentagon inside a rectangle. I colored in the pentagon and figured if I could subtract the area of the white space from the area of the rectangle, I'd have the area of the pentagon.



I know the area of the rectangle is $6 \times 7 = 42$ square units.

I saw that the white space was really five right triangles plus a little rectangle. The little rectangle is 1 by 2 units, so its area is $1 \times 2 = 2$ square units. Then, I figured the areas of the five right triangles: 1.5 square units, 1.5 square units, 3 square units, 3 square units, and 6 square units. So, the area of the white space is $2 + 1.5 + 1.5 + 3 + 3 + 6 = 17$ square units.

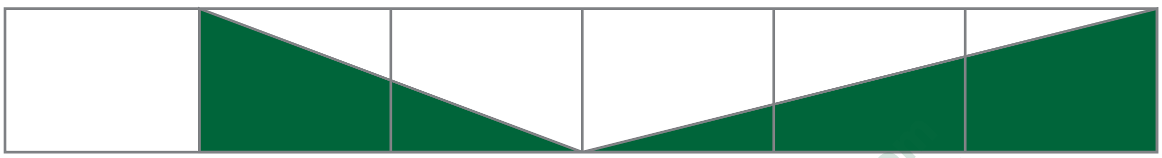
To get the area of the pentagon, I subtracted 17 from 42 and, like you, I got 25 square units for the area of the pentagon.

Rather than randomly calling on students to share their strategies, select students purposefully based on the strategies they have used to solve the problem. For instance, teachers can call on students who solved the problem using a strategy other than the one demonstrated, or a strategy not used by others in the class. Example 17 illustrates how two students might share their strategies for solving a fractions problem.

Example 17. Two students share their strategies for solving a fractions problem

Problem[116]

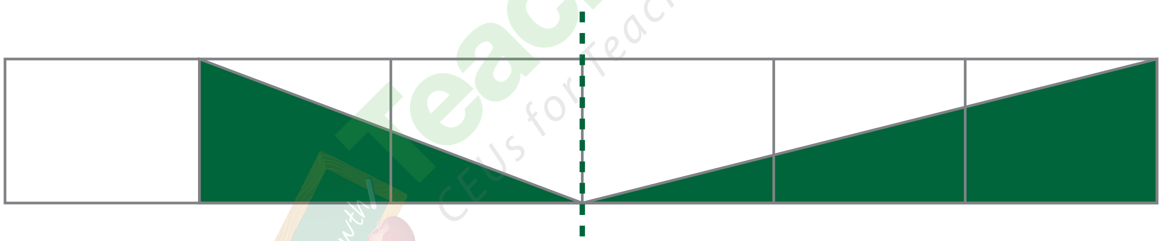
What fraction of the whole rectangle is green?



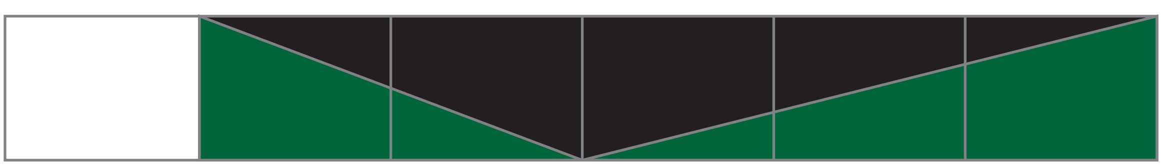
Solution strategies

STUDENT 1: If I think of it as what's to the left of the middle plus what's to the right of the middle, then I see that on the left, the green part is $\frac{1}{3}$ of the area; so that is $\frac{1}{3}$ of $\frac{1}{2}$ of the entire rectangle. On the right, the green part is $\frac{1}{2}$ of the area; so it is $\frac{1}{2}$ of $\frac{1}{2}$ of the entire rectangle. This information tells me that the green part is

$(\frac{1}{3} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) = \frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$ of the entire rectangle.



STUDENT 2: I see that the original green part and the part I've colored black have the same area. So the original green part is $\frac{1}{2}$ of the parts black and green, or $\frac{1}{2}$ of $\frac{5}{6}$ of the entire rectangle. This tells me that the green and black part is $\frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$ of the entire rectangle.



Ensure that students present not only their strategy but also an explanation for using the strategy. Engage students in a discussion about the specifics of their strategy by questioning them as they explain their thinking. For example, after the first student in Example 17 said, "I see that on the left, the green part is $\frac{1}{3}$," the teacher could ask, "How do you know?" The teacher also could ask, "How did you know that the green part on the right is half the area?" For the second student in Example 17, the teacher could ask, "How did you know that the green part is the same as the area colored black?"

Potential roadblocks and solutions

Roadblock 4.1. *Teachers don't have enough time in their math class for students to present and discuss multiple strategies.*

Suggested Approach. Teachers will need to purposefully select three to four strategies for sharing and discussing. To reduce the time students take to present their work to the class, ask them to use personal whiteboards or chart paper, or to bring their work to the document reader, so that they do not have to take time rewriting their strategies on the board. Teachers can document the strategies that different students use during independent or small-group work and summarize or display them for the whole class. This is likely to take less time than having students take turns sharing their own strategies. Teachers can also have students do a problem-solving task at the beginning of the school day as they settle in or at the beginning of the math class as a warm-up activity and devote 5–10 minutes to sharing and discussion.

Roadblock 4.2. *Not all students are willing to share their strategies.*

Suggested Approach. Fear of presenting wrong answers may limit the willingness of some students to share their strategies. It is important to create an environment in which students feel supported and encouraged to share, whether their strategy is correct or not. Teachers should emphasize that most problems can be solved using a variety of strategies and that each student may present a way to solve the problem that other students in the class had not thought to use. The panel believes that students will be more willing to explain their strategies once they notice that sharing helps them understand and solve problems better and gives students an opportunity to teach their classmates. Make sharing a regular part of mathematics instruction. Point out the benefits of sharing, such as how it enables students to teach their peers new ways to approach problems and to learn potentially quicker, easier, and more effective strategies from their peers.

Roadblock 4.3. *Some students struggle to learn multiple strategies.*

Suggested Approach. For some students who lack the prerequisite knowledge or do not remember the required skills, exposure to multiple strategies may be challenging.[117] For example, some students who do not remember their multiplication and division facts will have difficulty determining which strategy to use based on the numbers in a problem. Teachers may need to ask these students to take a minute and write down their facts before asking them to solve problems. Teachers also may need to change the problem to include numbers that allow these students to focus on the problem solving rather than on the arithmetic. For example, teachers could change the number 89.5 to 90. Some students may also get confused when multiple strategies for solving a problem are presented. Solving a problem one way, erasing it, and then solving it in another way can be difficult for students to comprehend if no opportunity is given to compare the methods side-by-side.[118] Students will also benefit from slowing down and having some time to get familiar with one strategy before being exposed to another or comparing it to a previously learned strategy.[119]

Roadblock 4.4. *Some of the strategies students share are not clear or do not make sense to the class.*

Suggested Approach. Students may have trouble articulating their strategies clearly so they make sense to the class. As students solve a problem, circulate among them and ask them to privately explain how they are working it. This will give teachers a clearer idea of the students' strategies. Then, when a student shares with the class, teachers will be better equipped to clarify the student's thinking by asking guiding questions or by carefully rewording what a student shares.[120] Another way to help students share is by asking another student to restate what the student has said.[121] Be careful not to evaluate what students share (e.g., "Yes, that is the right answer." or "No, that's not correct.") as students explain their thinking. Instead, ask students questions to help them explain their reasoning out loud (see Recommendation 2).

Help students recognize and articulate mathematical concepts and notation.

Mathematical concepts and notation provide students with familiar structures for organizing information in a problem; they also help students understand and think about the problem.[122] *When students have a strong understanding of mathematical concepts and notation, they are better able to recognize the mathematics present in the problem, extend their understanding to new problems,*[123] *and explore various options when solving problems.*[124] *Building from students' prior knowledge of mathematical concepts and notation is instrumental in developing problem-solving skills.*

In this recommendation, the panel suggests that teachers explain relevant concepts and notation in the context of a problem-solving activity, prompt students to describe how worked examples are solved using mathematically valid explanations, and introduce algebraic notation systematically. The panel believes these actions will help students develop new ways of reasoning, which in turn will help students successfully solve new mathematical challenges.

Summary of evidence: Moderate Evidence

Three studies directly support two suggestions for implementing this recommendation, and although the findings for the other suggestion are inconsistent, the panel believes there is moderate evidence supporting this recommendation.[125]

The first suggestion for implementing this recommendation, explaining relevant concepts and notation, was supported by a study finding that student achievement improved when teachers discussed math problems conceptually (without numbers) and then represented them visually.[126] Three studies examined the second suggestion for implementing this recommendation, providing students with worked examples and asking them to explain the process used to solve a problem; two studies reported positive effects, and one study reported no discernible effects.[127] Finally, two studies supported the third suggestion for implementing this recommendation, algebraic notation. The first study found that providing students with concrete intermediate arithmetic problems before asking them to understand the algebraic notation for a different problem significantly improved achievement.[128] The second study found that having students practice symbolic algebraic problems (substituting one expression into another) improved performance on two-step word problems more than practicing with one-step word problems.[129]

The panel identified three suggestions for how to carry out this recommendation.

How to carry out the recommendation

1. Describe relevant mathematical concepts and notation, and relate them to the problem-solving activity.

Students tend to enter school with informal, personally constructed ways of making sense of math.[130] Students often use this informal understanding to solve problems.[131] Teachers can turn problem-solving activities into learning opportunities by connecting students' intuitive understanding to formal mathematical concepts and notation.[132]

Teachers can watch and listen for opportunities to call attention to the mathematical concepts and notation that students use as they solve problems. For example, if teachers notice students informally using the commutative property to solve a problem, teachers can explain this concept, ask students if it will always work in similar situations, and describe the property's usefulness to the class.[133] If teachers see students talking about formal mathematical notation in an informal way, teachers can connect students' informal language to the formal notation or symbol, pointing out that there is often more than one mathematically correct way to state it (e.g., "12 take away 3", "12 minus 3", and "12 less 3" are all equal to 9). The teacher in Example 18 uses this technique to help her student better understand number theory. This example illustrates how students can better grasp formal mathematical concepts when teachers interpret the informal ideas and concepts students use to solve problems.[134]

Example 18. Students' intuitive understanding of formal mathematical concepts

Problem

Is the sum of two consecutive numbers always odd?

Solution

STUDENT: Yes.

TEACHER: How do you know?

STUDENT: Well, suppose you take a number, like 5. The next number is 6.

For 5, I can write five lines, like this:

| | | | |

For 6, I can write five lines and one more line next to it, like this:

| | | | | |

Then, I can count all of them, and I get 11 lines.

See? It's an odd number.

TEACHER: When you say, "It's an odd number," you mean the sum of the two consecutive numbers is odd. So, can you do that with any whole number, like *n*? What would the next number be?

STUDENT: It would be $n + 1$.

TEACHER: So, can you line them up like you did for 5 and 6?

STUDENT: You mean, like this?

n

$\underline{n + 1}$

TEACHER: Right. So, what does that tell you about the sum of n and $n + 1$?

STUDENT: It's 2 n's and 1, so it's odd.

TEACHER: Very good. The sum, which is $n + n + 1 = 2n + 1$, is always going to be odd.

Sometimes, teachers may need to draw attention to mathematical ideas and concepts by directly instructing students in them before engaging the students in problem solving.[135] For instance, some middle graders think of area as a product of numbers, rather than a measure of an array of units.[136] Faced with a problem such as the problem in Example 16 that asks them to find the area of a pentagon, students might measure the lengths of each side of the pentagon and calculate area by multiplying two or more of the numbers together. To offset this mistaken approach, their teacher might explicitly define *area* as "the measure of the space inside a figure."

2. Ask students to explain each step used to solve a problem in a worked example.

Routinely provide students with opportunities to explain the process used to solve a problem in a worked example and to explain why the steps worked.[137] Students will develop a better understanding of mathematical concepts when they are asked to explain the steps used to solve a problem in a worked example, and this understanding will help them solve problems successfully.[138] Studying worked examples could help accelerate learning and improve problem solving.[139]

Use small-group activities to encourage students to discuss the process used to solve a problem in a worked example and the reasoning for each step. Alternatively, ask one student to repeat another student's explanation and to then state whether he or she agrees and why.

Initially, students may not provide mathematically valid explanations. For example, students may restate the correct steps but fail to provide good reasons or justifications. The panel believes that teachers should ask students probing questions to help them articulate mathematically valid explanations. Mathematically valid explanations are factually and mathematically correct, logical, thorough, and convincing.[140] See Example 19 for sample student explanations that are *not* mathematically valid. Teachers also might need to provide students with examples of mathematically valid explanations or help reword their partially correct explanations. Example 19 also illustrates how teacher questioning can help students better organize their thoughts when their explanation is not mathematically valid.

Example 19. Sample student explanations: How mathematically valid are they?

Problem

Are $2/3$ and $8/12$ equivalent fractions? Why or why not?

An explanation that is not mathematically valid

STUDENT: To find an equivalent fraction, whatever we do to the top of $2/3$, we must do to the bottom.

This description is not mathematically valid because, using this rule, we might think we could add the same number to the numerator and denominator of a fraction and obtain an equivalent fraction. However, that is not true. For example, if we add 1 to both the numerator and denominator of $2/3$, we get $(2 + 1)/(3 + 1)$, which is $3/4$. $3/4$ and $2/3$ are not equivalent. Below is an explanation of how teacher questioning can clarify students' explanations and reasoning.

TEACHER: What do you mean?

STUDENT: It just works when you multiply.

TEACHER: What happens when you multiply in this step?

STUDENT: The fraction stays…the same.

TEACHER: That's right. When you multiply a numerator and denominator by the same number, you get an equivalent fraction. Why is that?

STUDENT: Before there were 3 parts, but we made 4 times as many parts, so now there are 12 parts.

TEACHER: Right, you had 2 parts of a whole of 3. Multiplying both by 4 gives you 8 parts of a whole of 12. That is the same part-whole relationship—the same fraction, as you said. Here's another way to look at it: when you multiply the fraction by $4/4$, you are multiplying it by a fraction equivalent to 1; this is the identify property of multiplication, and it means when you multiply anything by 1, the number stays the same.

A correct description, but still not a complete explanation

STUDENT: Whatever we multiply the top of $2/3$ by, we must also multiply the bottom by.

This rule is correct, but it doesn't explain why we get an equivalent fraction this way.

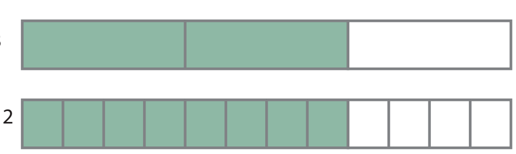
A mathematically valid explanation

STUDENT: You can get an equivalent fraction by multiplying the numerator and denominator of $2/3$ by the same number. If we multiply the numerator and denominator by 4, we get $8/12$.

If I divide each of the third pieces in the first fraction strip into 4 equal parts, then that makes 4 times as many parts that are shaded and 4 times as many parts in all. The 2 shaded parts become $2 \times 4 = 8$ smaller parts and the 3 total parts become $3 \times 4 = 12$ total smaller parts. So the shaded amount is $2/3$ of the strip, but it is also $8/12$ of the strip.

2/3

8/12



This explanation is correct, complete, and logical.

3. Help students make sense of algebraic notation.

Understanding the symbolic notation used in algebra takes time. The panel suggests that teachers introduce it early and at a moderate pace, allowing students enough time to become familiar and comfortable with it. Teachers should engage students in activities that facilitate understanding and better or correct use of symbols.[141] For instance, teachers can provide familiar arithmetic problems as an intermediate step before asking students to translate a problem into an algebraic equation (see Example 20).[142] Simple arithmetic problems draw on students' prior math experience, so the problems are more meaningful. By revisiting their earlier knowledge of simple arithmetic, students can connect what they already know (arithmetic) with new information (algebra).

Teachers also can ask students to explain each component of an algebraic equation by having them link the equation back to the problem they are solving (see Example 21).[144] This will help students understand how components in the equation and elements of the problem correspond, what each component of the equation means, and how useful algebra is for solving the problem.

Example 20. How to make sense of algebraic notation: Solve a problem arithmetically before solving it algebraically[143]

Problem

A plumbing company charges $42 per hour, plus $35 for the service call.

Solution

TEACHER: How much would you pay for a 3-hour service call?

STUDENT: $\$42 \times 3 + \$35 = \$161$ for a 3-hour service call.

TEACHER: What will the bill be for 4.5 hours?

STUDENT: $\$42 \times 4.5 + \$35 = \$224$ for 4.5 hours.

TEACHER: Now, I'd like you to assign a variable for the number of hours the company works and write an expression for the number of dollars required.

STUDENT: I'll choose h to represent the number of hours the company works.

$42h + 35 = \$$ required

TEACHER: What is the algebraic equation for the number of hours worked if the bill comes out to $140?

STUDENT: $42h + 35 = 140$

Example 21. How to make sense of algebraic notation: Link components of the equation to the problem[145]

Problem

Joseph earned money for selling 7 CDs and his old headphones. He sold the headphones for $10. He made $40.31. How much did he sell each CD for?

Solution

The teacher writes this equation:

$10 + 7x = 40.31$

TEACHER: If x represents the number of dollars he sold the CD for, what does the $7x$ represent in the problem? What does the 10 represent? What does the 40.31 represent? What does the $10 + 7x$ represent?

Potential roadblocks and solutions

Roadblock 5.1. *Students' explanations are too short and lack clarity and detail. It is difficult for teachers to identify which mathematical concepts they are using.*

Suggested Approach. Many students in the United States are not given regular opportunities to explain why steps to solve problems work;[146] without this experience to draw from, students who are suddenly given sharing opportunities often provide quick explanations that lack detail and clarity. They do not yet know how to explain the problem-solving process, which information to present, or how much detail to provide. To anticipate which mathematical concepts students might use to solve the problem, teachers may need to prepare for each lesson by solving the problem themselves.

To help students explain their thoughts in more detail, teachers can ask them specific questions about how a problem was solved and how they thought about the problem. Teachers can also have students create a "reason sheet" of mathematical rules (e.g., "the identity property of multiplication for fractions—multiplying a fraction by 1 keeps the same value," "$\frac{2}{2}$ and $\frac{3}{3}$ are fractions equal to 1," and so on). Students should only include a few key rules, as too many may make reasoning more difficult. Some helpful ones may appear in the student textbook as definitions, properties, rules, laws, or theorems. Ask students to use reasons from their reason sheet when composing explanations.

Roadblock 5.2. *Students may be confused by mathematical notations used in algebraic equations.*

Suggested Approach. Students may have difficulty interpreting mathematical notations used as variables in algebraic equations when the notations relate to items in the problem. Students may misinterpret the notations as labels for items in the problem (e.g., c stands for cookies). Teachers should encourage students to use arbitrary variables, such as non-mnemonic English letters (x and y) or Greek letters (Φ and Ω).[147] Arbitrary variables can facilitate student understanding of the abstract role that variables play in representing quantities.

The recommendations in this practice guide include research-based practices for teaching mathematical problem solving. Below is an example of how these recommendations can be incorporated into a lesson using a four-step process. It is important to consider all of the recommended practices when conducting a lesson from start to finish, even though you may not be able to apply all four steps for every problem-solving lesson.

1. Plan by **preparing appropriate problems and using them in whole-class instruction** *(Recommendation 1)* **and by selecting visual representations that are appropriate for students and the problems they are solving** *(Recommendation 3)*.

- **Include a variety of problem-solving activities** *(Recommendation 1).* Teachers should ask themselves, “Is the purpose of these problems to help students understand a key concept or operation? To help students learn to persist with solving difficult problems? To use a particular strategy? To use a visual representation?” Teachers should select problems that fit the goal of the lesson.

- **Ensure that students will understand the problem by addressing issues students might encounter with the problem’s context or language; consider students’ knowledge of mathematical content when planning the lesson** *(Recommendation 1).* Some problems may have complex or unusual vocabulary, depend on specific background knowledge, or reference ideas that are unfamiliar to some students. In some cases, it may be necessary to modify problems based on the learning or cultural needs of the students. In other cases, teachers may need to explain the context or language to students before asking them to solve the problem.

- **Select visual representations that are appropriate for students and the problems they are solving** *(Recommendation 3).* If the lesson will include a visual representation, teachers should consider the types of problems they plan to present and should select appropriate visual representations. Also, teachers should consider students’ past experience with visual representations to determine whether a new representation should be presented.

2. Depending on the content or goal of the lesson, **teach students how to use visual representations** *(Recommendation 3)*, **expose students to multiple problem-solving strategies** *(Recommendation 4)*, and/or **help students recognize and articulate mathematical concepts and notation** *(Recommendation 5)*.

If visual representations are featured in the lesson:

- **Use think-alouds and discussions to teach students how to represent problems visually** *(Recommendation 3).* Teachers should clarify the type of problem and how to determine which information in the problem is relevant. Teachers should then talk students through how to map the relevant information onto an appropriate visual representation and lead a discussion to compare representations. Allow students to share their work so that students can learn from others in the class. When students use their own representations, have them explain why and how they are using the representation. If any refinement is needed, build upon the student representation.

- **Show students how to convert the visually represented information into mathematical notation** *(Recommendation 3).* Teachers should demonstrate how each quantity and relationship in the visual representation corresponds to components of the equation.

If the goal is to teach students multiple strategies:

- **Provide instruction in multiple strategies** *(Recommendation 4).* Teachers should teach students a variety of ways to solve problems. These can be generic strategies that work for a wide range of problems or specific strategies that work for a given type of problem. Teachers can model the use of strategies by thinking aloud about why they selected the particular strategy and how they would work a problem.

- **Ask students to generate multiple strategies for solving a problem** *(Recommendation 4).* Encourage students to generate strategies of their own as they work through the problems they are given.

If the goal is to help students recognize and articulate mathematical concepts and notation:

- **Describe relevant mathematical concepts and notation, and relate them to the problem-solving activity** *(Recommendation 5).* Teachers should look for opportunities to explain the formal mathematical concepts and notation used in the problem-solving activity.

- **Help students make sense of algebraic notation** *(Recommendation 5).* One way to do this is to introduce similar arithmetic problems before algebraic problems to revisit students' earlier mathematical understanding. Another way is to help students explain how the algebraic notation represents each component in the problem.

3. **Assist students in monitoring and reflecting on the problem-solving process** *(Recommendation 2).*

- **Provide students with a list of prompts to help them monitor and reflect during the problem-solving process** *(Recommendation 2).* It may be necessary to assist students as they begin to work with prompts. Assisting means more than simply telling students what to do next. Teachers can assist by asking students guiding questions to help them learn to use the prompts when they solve problems.

- **Model how to monitor and reflect on the problem-solving process** *(Recommendation 2).* Teachers can state a prompt in front of the class and describe how they used it to solve a problem. This will help students see how prompts or items from a task list can be used to solve problems. Teacher modeling is a useful way to show how people think as they solve problems.

- **Use student thinking about a problem to develop students' ability to monitor and reflect on their thought process while solving a problem** *(Recommendation 2).* Teachers can ask guiding questions to help students verbalize what they could do to improve their monitoring and reflection during the problem-solving process.

4. Conduct discussions **to help students recognize and articulate mathematical concepts and notation** *(Recommendation 5)* and to **expose students to multiple problem-solving strategies** *(Recommendation 4).*

- **Ask students to explain each step used to solve a problem** *(Recommendation 5).* Debriefing each step allows teachers to connect the problem-solving activity to relevant mathematical concepts and notation.

- **Provide opportunities for students to compare multiple strategies in worked examples; ask students to generate, share, and compare multiple strategies for solving a problem** *(Recommendation 4).* This approach allows students to hear multiple problem-solving strategies, which is particularly beneficial if the strategies are more advanced than those used by most students in the classroom. It also affords students the chance to present and discuss their strategies, thereby building their confidence levels.

Rationale for Evidence Ratings[a]

Appendix D provides further detail about studies that the panel used to determine the evidence base for the five recommendations in this guide. Studies that examined the effectiveness of recommended practices using strong designs for addressing questions of causal inference including randomized controlled trials and rigorous quasi-experimental designs and that met What Works Clearinghouse (WWC) standards (with or without reservations) were used to determine the level of evidence and are discussed here. This appendix also includes one study with a strong correlational design.[148]

Four studies met the WWC pilot standards for well-designed single-case design research and are included as supplemental evidence for Recommendations 2 and 3 in this guide. Single-case design studies do not contribute to the level of evidence rating. While the panel believes that qualitative studies, case studies, and other correlational studies contribute to the literature, these studies were not eligible for WWC review, did not affect the level of evidence, and are not included in this appendix. Some studies have multiple intervention groups; only interventions relevant to this guide's recommendations are included.[149]

In this practice guide, a group-design study[150] result is classified as having a positive or negative effect when:

- The result is statistically significant ($p \leq 0.05$) or marginally statistically significant ($0.05 < p \leq 0.10$)
- The result is substantively important as defined by the WWC (effect sizes larger than 0.25 or less than –0.25)[151]

When a result meets none of these criteria, it is classified as having "no discernible effect."

Some studies meet WWC standards (with or without reservations) for causal designs but do not adjust statistical significance for multiple comparisons or student clusters where the unit of assignment is different from the unit of analysis (e.g., classrooms are assigned to conditions, but student test scores are analyzed). When full information is available, the WWC adjusts for clustering and multiple comparisons within a domain.[152]

The three outcome domains[153] for this practice guide are as follows:

- **Procedural knowledge**, which relates to whether students choose mathematical operations and procedures that will help them solve the problem and to how well they carry out the operations and procedures they choose to use. When students correctly solve a math problem, they have likely chosen the appropriate operation or procedure and executed it correctly.
- **Conceptual understanding**, which relates to how well students understand mathematical ideas, operations, and procedures, as well as the language of mathematics. One way for students to express their conceptual understanding is to accurately and completely explain the operations and ideas used to solve a problem. Another way to show conceptual understanding is for students to explain relationships between ideas, operations, and/or procedures as they relate to a problem.
- **Procedural flexibility**, which relates to whether students can identify and carry out multiple methods to solve math problems. If students can adaptively choose the most appropriate strategy for a particular problem and can attempt to solve a math problem in multiple ways, then they have likely developed procedural flexibility, a skill that may help them solve problems more efficiently in the future.

[a] Eligible studies that meet WWC evidence standards or meet evidence standards with reservations are indicated by **bold text** in the endnotes and references pages.

Most studies only examined outcomes in the procedural knowledge domain, and thus *student achievement* in this appendix refers to outcomes in the procedural knowledge domain except where otherwise specified. To facilitate comparisons, the appendix text focuses on the outcome closest to the end of the intervention; these are labeled *posttests*. All outcome measures administered after the posttest are labeled *maintenance* in appendix tables. Measures the panel believes require students to apply knowledge or skills in a new context are labeled *transfer outcomes* in appendix tables. When studies have multiple posttest outcome measures administered within the same domain, effect sizes for each measure are averaged,[154] and the overall average is reported.

Recommendation 1. Prepare problems and use them in whole-class instruction.

Level of evidence: Minimal Evidence

Few studies directly tested the suggestions of this recommendation, leading the panel to assign a minimal level of evidence. Although the panel believes teacher planning should incorporate both routine and non-routine problems, no studies meeting WWC standards directly examined this issue.

One study did find higher student achievement when teacher planning considered students' mathematical content weaknesses and whether students would understand language and context prior to instruction (see Table D.1).[155] Another study showed that incorporating a variety of familiar contexts into instruction also may improve problem-solving skills.[156] The panel interpreted these results cautiously, however, since both of these studies included additional instructional components (e.g., student monitoring and reflection). The panel did find several well-designed studies showing that Taiwanese and American students who solve word problems incorporating well-liked and well-known contexts do better on subsequent word problems tests than students presented with generic contexts.[157]

Routine and non-routine problems. The panel's suggestion to integrate a variety of targeted problem-solving activities into whole-class instruction is based primarily on the expertise of its members. No studies meeting WWC standards directly tested the complementary uses of routine and non-routine problems.

Problem context and vocabulary. Overall, no studies included interventions that solely tested this recommendation suggestion. One study meeting WWC standards involved teacher planning that considered whether students would have difficulty understanding the language, context, or mathematical content in word problems (see Table D.1).[158] Teachers also reviewed the problems' vocabulary with students during instruction. This planning and instruction were part of a broader intervention that also involved teaching students to pose questions to themselves while problem solving. The overall intervention had a significant positive effect on students' ability to solve word problems.

Another study involved word problems that incorporated contexts familiar to the 5th-grade students in the study.[159] However, these contexts were only one component of an intervention that also focused on teaching students a five-step strategy for solving word problems. Although the study reported a positive effect, the panel cannot isolate the separate effect of posing familiar word problems.

Four additional studies found that incorporating favorite contexts into practice word problems improved students' ability to solve multiplication and division word problems.[160] Three of the studies were conducted in Taiwan with 4th- and 5th-grade students, and one took place in the United States with students in grades 6 through 8. None of these studies had transfer or maintenance outcomes. In all of the studies, researchers asked students about their favorite places, foods, friends, sports, and so on, and then incorporated that information into the practice problems. In two of the studies, word problems used during classroom instruction were based on the most common survey responses.[161] In the other two, students completed computer-based word problems that incorporated their individual survey responses.[162]

These favorite-context interventions were conducted in two to four sessions, each lasting between 40 and 50 minutes. Students in the control group received the same word problems but without the personalized content. Three of the four studies found that personalizing the content of word problems improved students' subsequent performance on a posttest that included both personalized and nonpersonalized word problems.[163] In the study that found no discernible effects, the panel believes the outcome measure limited the study's ability to detect differences between groups.

Planning using students' math knowledge. Teachers in the study on teacher vocabulary planning[164] also used information from earlier student performances to help them understand and plan for difficulties with mathematical content that students might have. The overall intervention had a positive effect on students' ability to solve word problems.

Table D.1. Studies of interventions that involved problem selection and contribute to the level of evidence rating

Study	Comparison	Duration	Students	Math Content	Outcomes[165]	Effect Size
Familiar Contexts in Problems						
Verschaffel et al. (1999) *Quasi-experimental design*	Word problems with familiar contexts for students[166] vs. traditional instruction and standard textbook problems	A total of 20 sessions, each lasting 50–60 minutes	A total of 203 students in the 5th grade in Belgium	Word problems involving numbers and operations	Posttest (average of a posttest and a retention test)[167]	0.31**[168]
				General math achievement	Transfer	0.38**[169]
Preferred Contexts in Problems						
Chen and Liu (2007) *Randomized controlled trial*	Word problems featuring students' preferences vs. standard textbook problems	A total of four sessions, each lasting 50 minutes	A total of 165 students in the 4th grade in Taiwan	Word problems involving numbers and operations	Posttest	0.72**
Ku and Sullivan (2000) *Randomized controlled trial*	Word problems featuring students' preferences vs. standard textbook problems	A total of two sessions, each lasting 50 minutes	A total of 72 students in the 5th grade in Taiwan	Word problems involving numbers and operations (multiplication and division)	Posttest (average of subtests with personalized and nonpersonalized problems)	0.13, ns
Ku and Sullivan (2002) *Randomized controlled trial*	Word problems featuring students' preferences vs. standard textbook problems	A total of three sessions, each lasting 40 minutes	A total of 136 students in the 4th grade in Taiwan	Word problems involving numbers and operations	Posttest (average of subtests with personalized and nonpersonalized problems)	0.34*
Ku et al. (2007) *Randomized controlled trial*	Word problems featuring students' preferences vs. standard textbook problems	A total of two sessions, each lasting 42 minutes	A total of 104 students in grades 6–8 in the United States	Word problems involving numbers and operations	Posttest	0.28, ns
Clarifying Vocabulary and Math Content						
Cardelle-Elawar (1995) *Randomized controlled trial*	Teacher consideration for whether students would understand problems and review of vocabulary and math content with students[170] vs. traditional instruction	One year	A total of 463 students in grades 4–8 in the United States[171]	Word problems involving general math achievement	Posttest (average of posttest and two retention tests administered over seven months)[172]	2.18**

** = statistically significant at 0.05 level
* = statistically significant at 0.10 level
ns = not statistically significant

Recommendation 2.
Assist students in monitoring and reflecting on the problem-solving process.

Level of evidence: Strong Evidence

The panel assigned a rating of strong evidence to this recommendation based on nine studies that met WWC standards with or without reservations (see Table D.2).[173] All nine studies reported positive effects on students' ability to solve word problems. The outcomes measured diverse mathematical content, including numbers and operations, data analysis and probability, geometry, and algebra. Researchers conducted the studies in 4th- through 8th-grade classrooms, with three of the studies taking place in countries outside the United States. All the interventions taught students to monitor and structure their problem-solving process, although the specific prompts varied. Four studies provided students with a list of key tasks for solving word problems,[174] while teachers in the other five studies taught students to use self-questioning and reflection.[175] Two of these studies combined interventions—a task list along with visual aids—so the panel could not attribute its results solely to the task checklist.[176]

Prompting students with lists. Several studies, including some that also involved teacher modeling, prompted students to self-question or to complete tasks or steps while problem solving. In two studies, teachers guided the self-questioning process by including questions in students' practice workbooks.[177] Students would answer the questions verbally and then in writing when using their workbooks.[178] In another study, students received index cards with question prompts and then asked each other questions while working in pairs to solve word problems in a commercial software program.[179] In all of these studies, the intervention's effects were positive.

Other studies examined using task lists to prompt students. In one study with 5th-grade students in Belgium, teachers discussed strategies for solving word problems across 20 lessons in four months.[180] Teachers combined whole-class instruction with small-group work and individual assignments. The control group received traditional instruction for word problems, which did not include using a task list. Results from the study showed that the intervention had a positive effect on students' ability to solve word problems.

In two other studies, students received checklists with four-step strategies and were encouraged to think aloud while solving problems independently.[181] The interventions in these two studies were very similar and included the use of both schematic diagrams and other non-prompting components. Comparison teachers used district-adopted textbooks that focused on direct instruction, worked examples, and student practice. Instruction in the first study occurred during 10 classes, each 40 minutes long. The intervention had a positive effect on 7th-grade students' ability to solve ratio and proportion word problems, both immediately afterward and four months later.[182] Instruction in the second study occurred during 29 classes, each 50 minutes long. The intervention had a positive effect on 7th-grade students' ability to solve ratio, proportion, and percent word problems immediately afterward. There was no discernible effect on an identical test given one month after the intervention ended or on a transfer test.[183]

In a fourth study, 4th- and 5th-grade students received a five-step strategy list.[184] Teachers demonstrated how to use the strategy and then asked students to complete two practice problems individually. Finally, students discussed in pairs how they implemented the strategy. Comparison students solved the same problems individually and in groups, but without the five-step strategy. The study found that use of the checklist strategy improved students' performance on a four-item test of word problems measuring general math achievement.

Modeling monitoring and reflection. Five randomized controlled trials and quasi-experimental studies examined how self-questioning could help students monitor, reflect, and structure their problem solving.[185] In these studies, teachers modeled the self-questioning process and taught students how to ask themselves questions while problem solving.[186] For example, in one study, teachers modeled the self-questioning process by reading the questions and verbalizing their thoughts aloud.[187] Compared to students who completed the same problems but were not told to question one another, students in this intervention experienced positive effects on their ability to solve geometry word problems. Similarly, four studies found positive results when teachers modeled multistep strategies using task lists and then asked students to structure their problem solving around those strategies.[188]

In two of the self-questioning studies, teachers modeled desired problem solving by asking and answering questions derived from a problem-solving model.[189] Teachers also were taught to practice self-questioning when preparing lessons (e.g., asking, "What are the key errors students might make?"). These two studies provided similar instruction but differed in student samples and duration: one study involved six hours of oral feedback over three weeks to low-achieving 6th-grade bilingual students,[190] and the other study followed students in grades 4 through 8, including several bilingual classes, and took place over the course of an entire school year.[191] Both studies found positive effects on students' ability to solve word problems compared to students who received only traditional instruction with lectures and worksheets.

Two additional studies in Israel had teachers model a self-questioning approach (the IMPROVE method) for the whole class. Teachers instructed 8th-grade students to ask themselves four types of questions while solving word problems: (1) comprehension questions, to ensure they understood the task or concept in the problem; (2) connection questions, to think through similarities between problem types; (3) strategic questions, to focus on how to tackle the problem; and (4) reflection questions, to think about what they wanted to do during the solution process.[192] Comparison conditions differed for the studies: in one, students received worked examples followed by practice problems,[193] while in the other, instruction involved whole-class lectures and practice problems. Both studies found a positive effect of teaching and actively supporting students to use the questions.[194]

Supplemental evidence comes from three single-case design studies. The first study, involving 3rd- and 4th-grade students, found that teacher modeling of a self-questioning approach improved achievement for students with learning disabilities or mild intellectual disabilities.[195] In this study, students were first taught a nine-step problem-solving strategy, and the instructor and student discussed the importance of self-questioning. After the students generated statements applying the strategy, the instructor and student then modeled the self-questioning process. The two other single-case design studies found no evidence of positive effects.[196] However, in one study, students were already achieving near the maximum score during baseline, and thus the outcome could not measure any improvement.[197] In the other study, middle-school students with learning disabilities were taught a seven-step self-questioning process. Based on the findings reported, there is no evidence that this intervention had a positive impact on student achievement.

Table D.2. Studies of interventions that involved monitoring and contribute to the level of evidence rating

Study	Comparison	Duration	Students	Math Content	Outcomes[198]	Effect Size
Cardelle-Elawar (1990) *Randomized controlled trial*	Instruction in monitoring and reflecting using questions vs. traditional instruction	Six hours	A total of 80 low-achieving 6th-grade students from bilingual classes in the United States	Word problems involving general math achievement	Posttest	2.54**[199]
Cardelle-Elawar (1995) *Randomized controlled trial*	Instruction in monitoring and reflecting using questions vs. traditional instruction	One school year	A total of 463 students in grades 4–8 in the United States[200]	Word problems involving general math achievement	Posttest (average of a posttest and two retention tests given over seven months)[201]	2.18**
Hohn and Frey (2002) *Randomized controlled trial*	Instruction in monitoring and reflecting using a task list vs. no instruction in monitoring and reflecting	A total of four sessions presented every two days	A total of 72 students in the 4th and 5th grades (location not reported)[202]	Word problems involving general math achievement	Posttest	0.79, ns
Jitendra et al. (2009) *Randomized controlled trial*	Instruction in monitoring and reflecting using questions and a task list[203] vs. traditional instruction	A total of 10 sessions, each lasting 40 minutes	A total of 148 students in the 7th grade in the United States	Word problems involving numbers and operations	Posttest	0.33, ns
					Maintenance (four months after posttest)	0.38, ns
				State assesment	Transfer	0.08, ns
Jitendra et al. (2010) *Randomized controlled trial*	Instruction in monitoring and reflecting using questions and a task list[204] vs. traditional instruction	A total of 29 sessions, each lasting 50 minutes	A total of 472 students in the 7th grade in the United States	Word problems involving numbers and operations	Posttest	0.21**
					Maintenance (one month after posttest)	0.09, ns
					Transfer	–0.01, ns
King (1991) *Randomized controlled trial with high attrition and baseline equivalence*	Instruction in monitoring and reflecting using questions vs. no instruction in monitoring or reflecting	A total of six sessions, each lasting 45 minutes, across three weeks	A total of 30 students in the 5th grade in the United States	Word problems and problem solving involving geometry	Posttest	0.98*[205]
Kramarski and Mevarech (2003) *Randomized controlled trial with unknown attrition and baseline equivalence*	Instruction in monitoring and reflecting using questions vs. no instruction in monitoring and reflecting	A total of 10 sessions, each lasting 45 minutes	A total of 384 students in the 8th grade in Israel	Multiple-choice problems and word problems involving data analysis	Posttest	0.48
				Data analysis	Posttest (flexibility competency)	0.77
				Data analysis	Transfer (graphical representations)	0.37
Mevarech and Kramarski (2003) *Randomized controlled trial*	Instruction in monitoring and reflecting using questions vs. instruction that had students study worked examples and then discuss their solutions to problems	Four weeks	A total of 122 students in the 8th grade in Israel	Word problems involving algebra	Posttest	0.34, ns
					Maintenance (one year after posttest)	0.31, ns

(continued)

Table D.2. Studies of interventions that involved monitoring and contribute to the level of evidence rating *(continued)*

Study	Comparison	Duration	Students	Math Content	Outcomes[198]	Effect Size
Verschaffel et al. (1999) *Quasi-experimental design*	Instruction in monitoring and reflecting using a task list vs. traditional instruction	A total of 20 sessions, each lasting 50–60 minutes	A total of 203 students in the 5th grade in Belgium	Word problems involving numbers and operations	Posttest (average of a posttest and a retention test)[206]	0.31**[207]
				General math achievement	Transfer	0.38**[208]

** = statistically significant at 0.05 level
* = statistically significant at 0.10 level
ns = not statistically significant

Table D.3. Supplemental evidence supporting the effectiveness of Recommendation 2

Study	Comparison	Duration	Students	Math Content	Outcomes[209]	Effect Size[210]
Cassel and Reid (1996) *Single-case design*	Instruction in monitoring and reflecting using questions and a task list[211] vs. no instruction	Unknown number of sessions, each lasting 35 minutes	Four 3rd and 4th grade students with mild mental handicaps in the United States	Word problems involving numbers and operations	Repeated measurement	Positive evidence
Case et al. (1992) *Single-case design*	Instruction in monitoring and reflecting using a task list[212] vs. no instruction	Between four and five sessions, each lasting about 35 minutes	Four 5th and 6th grade students with learning disabilities in the United States	Word problems involving numbers and operations	Repeated measurement	No evidence
Montague (1992) *Single-case design*	Instruction in monitoring and reflecting using a task list[213] vs. no instruction	A total of three sessions, each lasting 55 minutes	Three students with learning disabilities in grades 6–8 in the United States	Word problems involving numbers and operations	Repeated measurement	No evidence

Recommendation 3. Teach students how to use visual representations.

Level of evidence: Strong Evidence

The panel determined there is strong evidence supporting this recommendation; several studies with diverse student samples—mostly taking place in the United States, with middle school students—consistently found that using visual representations improved problem-solving achievement (see Table D.3).[214]

Both general-education students and students with disabilities performed better when specifically taught how to use different visual representations for different types of problems:[215] for example, to identify and map relevant information onto schematic diagrams[216] and to integrate visual representations.[217] One study further suggested that student achievement increases more when students learn how to design, develop, and improve their own representations than when students use teacher- or textbook-developed visuals.[218]

Selecting visuals.[219] Several studies consistently found positive results when students were taught to use visual representations to solve problems.[220] For example, four studies taught students to solve numbers-and-operations word problems with only schematic diagrams.[221] In another study, the authors taught 7th-grade students to use a linking table to overcome students' mistaken intuitive beliefs about multiplication and division.[222] Each study found positive effects compared to students who received traditional instruction.

Using visuals. Multiple studies with positive results involved teachers providing different types of visual representations for different types of problems.[223] For example, in one study, middle school students received papers listing the prominent features of two different kinds of problems (e.g., for proportion problems, "There are two pairs of associations between two things that involve four quantities").[224] Students then used type-specific diagrams to represent these problems. Initially, student worksheets included just one kind of problem, but after students learned how to solve both, worksheets with multiple problem types were presented, and students could compare them. Students receiving this instruction scored higher than comparison students who were taught using the textbook. Teachers in the comparison condition also modeled how to use representations to represent information in problems.

In another study, students practiced identifying different problem types and then mapping the features of each on a schematic diagram.[225] Practice problems were grouped by problem type, with students identifying the critical elements. Teachers also repeated explicit instructions in order to provide strategy steps and clarify misconceptions. Students receiving this intervention had higher achievement than students in the comparison group on both a posttest conducted one to two weeks after the intervention and a transfer test that used problems taken from a textbook. These positive effects persisted for four months after the intervention; however, the intervention had no discernible effects on a state standardized test.

Supplemental evidence comes from one single-case design study in which students were taught how to use visual representations. The study found no evidence of an effect.[226] Teachers demonstrated the diagram-mapping process and instructed 6th- and 7th-grade students with learning disabilities on how to represent the diagram information as a mathematical sentence. Students were taught to identify different types of addition and subtraction problems.[227]

Translating visuals. In two studies involving students with learning disabilities or mild disabilities, students were taught to diagram story situations that contained all necessary information. Later, teachers presented these students with word problems and asked them to represent unknown information with question marks.[228] In one of the studies, instruction emphasized that the ultimate mathematical equation could be derived directly from the word problem diagram.[229] Similarly, instruction in the other study suggested that the mathematical equation could be identified directly from a data table.[230] Both studies found positive effects.

Table D.4. Studies of interventions that used visual representations and contribute to the level of evidence rating

Study	Comparison	Duration	Students	Math Content	Outcomes[231]	Effect Size
Jitendra et al. (1998) *Randomized controlled trial*	Instruction in the use of a visual representation (schematic drawing) vs. traditional instruction	Between 17 and 20 sessions, each lasting 40–45 minutes	A total of 34 students in grades 2–5 in the United States[232] (most students had mild disabilities)	Word problems involving numbers and operations	Posttest	0.56, ns
					Maintenance (one to two weeks after intervention)	0.85**
					Transfer (one day after posttest)	1.01**
Jitendra et al. (2009) *Randomized controlled trial*	Instruction in the use of a visual representation (schematic drawing)[233] vs. traditional instruction	A total of 10 sessions, each lasting 40 minutes	A total of 148 students in the 7th grade in the United States	Word problems involving numbers and operations	Posttest	0.33, ns
					Maintenance (four months after posttest)	0.38, ns
				State assessment	Transfer	0.08, ns
Jitendra et al. (2010) *Randomized controlled trial*	Instruction in the use of a visual representation (schematic drawing)[234] vs. traditional instruction	A total of 29 sessions, each lasting 50 minutes	A total of 472 students in the 7th grade in the United States	Word problems involving numbers and operations	Posttest	0.21**
					Maintenance (one month after posttest)	0.09, ns
					Transfer	–0.01, ns
Selke et al. (1991) *Randomized controlled trial*	Instruction in the use of a visual representation (data table) vs. instruction in a substitution strategy	A total of 10 sessions during regular math class	A total of 107 students in the 7th grade in the United States	Word problems involving numbers and operations[235]	Posttest (average of two subtests)	1.29**
Xin et al. (2005) *Randomized controlled trial*	Instruction in the use of a visual representation (schematic drawing) vs. traditional instruction, including the use of a semi-concrete representation to represent information	A total of 12 sessions, each lasting one hour	A total of 22 students in grades 6–8 in the United States (most students had learning disabilities)	Word problems involving numbers and operations	Posttest	1.87**
					Maintenance (administered 3–12 weeks after intervention)[236]	3.03**
					Transfer	1.51**
			Student-Generated Visual Representations			
Terwel et al. (2009) *Randomized controlled trial*	Instruction to students on generating their own visuals vs. providing students with teacher-made visuals	A total of 13 sessions, each lasting one hour, across three weeks	A total of 238 students in the 5th grade in the Netherlands	Word problems involving data analysis and probability	Posttest	0.41, ns
					Transfer	0.64**

** = statistically significant at 0.05 level
* = statistically significant at 0.10 level
ns = not statistically significant

Table D.5. Supplemental evidence supporting the effectiveness of Recommendation 3

Study	Comparison	Duration	Students	Math Content[237]	Outcome	Effect Size[238]
Jitendra et al. (1999) *Single-case design*	Instruction in the use of a visual representation (schematic diagram)[239] vs. no instruction	Unknown number of sessions, each lasting 45 minutes	Four 6th- and 7th-grade students with learning disabilities in the United States	Word problems involving numbers and operations	Repeated measurement	No evidence

Recommendation 4. Expose students to multiple problem-solving strategies.

Level of evidence: Moderate Evidence

Eight studies found positive effects of teaching and encouraging multiple problem-solving strategies, although in some studies, the effects were not consistent across all types of outcomes.[240] Three studies involving students with no or limited knowledge of algebra found negative effects for some algebra outcomes, but these students had not developed sufficient skills in a domain before multiple-strategies instruction. Consequently, the panel determined that there is moderate evidence to support this recommendation. Six of the seven studies that examined procedural flexibility—the ability to apply multiple problem-solving approaches—found that teaching multiple problem-solving strategies, either by instruction, worked examples, or prompting, improved students' procedural flexibility (see Table D.4).[241] Yet teaching multiple strategies had mixed effects on students' procedural knowledge, or ability to solve problems correctly, with five studies reporting positive effects on posttests, two studies reporting no discernible effects, and the three studies involving students with no or minimal algebra knowledge reporting negative effects on algebra procedural knowledge outcomes.[242] Effects on conceptual knowledge of mathematics were inconsistent, with two studies finding no discernible effects, one study finding positive effects, and one study finding that the effects depended on baseline knowledge in the domain.[243]

Multiple-strategies instruction. Two studies involving multiple-strategies instruction found positive effects on procedural knowledge; however, because both of these interventions incorporated multiple components, the panel could not attribute their results solely to multiple-strategies instruction. One of the studies included instruction that emphasized different solution strategies.[244] Teachers directly taught a variety of solution methods—unit-rate strategies, cross multiplication, and equivalent fractions—to solve ratio-and-proportion word problems, and students learned to identify when a particular strategy was appropriate. Seventh-grade students who received this instruction for 10 sessions made greater achievement gains than students who received traditional lessons including direct instruction, worked examples, and guided practice. The second study used a similar intervention but with more instruction time and professional development.[245]

In another study, 8th-grade students in Germany were taught to work forward and backward, as needed, to solve problems during an after-school tutoring program.[246] These students performed significantly better than students who received no after-school tutoring. However, because the comparison students received no organized instruction, the panel is unsure whether this result applies to classroom settings.

A final study involving received a brief, eight-minute period of strategy instruction.[247] An instructor solved three equations on a blackboard using different strategies. For each equation, the instructor used the strategy that led to the most efficient solution to each problem. Students were not told why a particular strategy was selected. Prior to this instruction, none of the participants had received formal instruction on equation solving. There were no discernable effects for this brief intervention.

Worked examples. Three of the four studies examining this intervention found that teaching students to compare multiple strategies on worked examples improved procedural flexibility—but these studies also found that the effects on procedural and conceptual knowledge were sometimes positive and sometimes not discernible.[248] In each of these studies, students worked in pairs, and the researchers manipulated only how the worked examples were presented. Both intervention and control students were exposed to multiple solution strategies; however, the comparison of the multiple solution strategies was only facilitated for intervention students.

For example, all students in one study reviewed packets of worked examples, in which half the worked examples presented the conventional solution method and half presented a shortcut solution method.[249] On each page, two worked examples were presented side-by-side as a pair. For the intervention group, each worked-example pair contained the same equation, solved using both the conventional and shortcut strategies. In the control group, each worked-example pair contained two different problems solved with the same solution strategy. Thus, only the intervention condition facilitated comparisons between different strategies. Students were also presented with practice problems. In the intervention condition, students were asked to solve two practice problems using two different strategies for each, while control students received four different equations and were not asked to use different strategies. Intervention students scored higher than control students on measures of conceptual knowledge and procedural flexibility, and this impact persisted for two weeks after the intervention ended. However, there were no discernible differences between groups on procedural knowledge.

The other two studies facilitated multiple-strategy comparison for intervention students by providing worked-example packets with each worked example solved in two different ways on the same page; control students received the same number of worked examples, but each of the problems was different and presented on a *separate* page.[250] In the first study, 7th-grade students solved algebra problems; this study found that facilitating multiple-strategies comparison improved both procedural knowledge and procedural flexibility, but that there was no impact on conceptual knowledge. The second study involved multiplication estimation problems and found no discernible effects on procedural or conceptual knowledge, either immediately or after two weeks; it did, however, find a positive effect on procedural flexibility that persisted for two weeks after the intervention ended.

The fourth study had a similar intervention, but the participants were students with no or limited pretest algebra knowledge.[251] Specifically, the study involved two groups of students: students who never used an algebraic problem-solving approach on a pretest, and students who attempted an algebraic approach, correctly or incorrectly.[252] (Even the second group of students had limited algebra problem-solving skills—roughly two-thirds used algebra incorrectly on the pretest.) All participants were 7th- and 8th-grade students at a low-performing middle school. The intervention facilitated multiple-strategy comparison for intervention students by providing worked-example packets with each example solved in two different ways on the same page, while control students received packets with the worked examples on a different page. Additionally, at the end of the three daily sessions, intervention students were presented with two problems and asked to solve each of the problems using two different solution methods, while control students were presented with four problems and allowed to choose their solution method. For the students who did not attempt algebraic problem solving on the pretest, the study found the intervention had negative effects on procedural knowledge, conceptual knowledge, and procedural flexibility. However, there were no discernible effects on any of these outcomes for students who attempted an algebraic approach on the pretest.[253]

Generating and sharing multiple strategies. Four studies, with one study involving two comparisons, examined this approach, with three comparisons focusing on procedural knowledge and finding mixed effects, and three comparisons focusing on procedural flexibility and finding positive effects.

Of the three studies that examined procedural knowledge, one study found a positive effect and two studies found negative effects.[254] The panel believes these different findings result from important differences in the student samples and how students were prompted to generate and share multiple strategies.[255] In the study with positive results, there are two comparisons related to this recommendation: one involved individual students generating multiple strategies, and the other considered pairs of students collaborating to generate multiple strategies.[256] Students in both groups also shared strategies and answers after solving an initial problem, and they also used instructional components such as manipulatives. In the comparison group, multiple strategies were not discussed, and students completed their work individually after being provided with solution steps. Although both interventions had positive effects on procedural knowledge, the effect size for the two-person group comparison was about twice as large. Pretest scores indicate that the 3rd- and 4th-grade student sample had some relevant problem-solving knowledge, even though participants scored in the lower half of pretest achievement.

In the two studies with negative findings on procedural knowledge, participants did not have basic algebra knowledge, and the interventions involved algebra. In the first study, intervention students were prompted to generate multiple strategies by simply reordering problem-solving steps, while comparison students were not prompted.[257] No teacher instruction or student sharing took place. None of the participants had received formal instruction on equation solving in school. Participants in the second study were 6th-graders with no baseline algebra knowledge who received a 30-minute lesson on equation solving and then practiced solving algebra problems.[258] Intervention students were given algebra problems they had previously solved and were asked to re-solve them using a different ordering of steps, while comparison students were not given instructions but were provided with additional, similar problems. The panel noted that both studies found positive effects on procedural flexibility—a more aligned outcome.

This procedural-flexibility finding was supported by another study, this one involving 8- and 9-year-olds in the United Kingdom.[259] Working on computers, intervention students were provided with a monetary amount (represented with coins) and asked to develop other combinations of coins that would total the same amount. These students scored significantly higher on procedural flexibility than comparison students, who did not use the computer program and were not asked to generate multiple combinations.

Table D.6. Studies of interventions that involved multiple problem-solving strategies and contribute to the level of evidence rating

Study	Comparison	Duration	Students	Math Content	Domain and Outcome	Effect Size
			Instruction in Multiple Strategies			
Jitendra et al. (2009) *Randomized controlled trial*	Instruction in problem-specific multiple strategies[260] vs. traditional instruction	A total of 10 daily sessions, each lasting 40 minutes	A total of 148 students in the 7th grade in the United States	Word problems involving numbers and operations	Procedural posttest	0.33, ns
					Procedural maintenance (four months after posttest)	0.38, ns
				State assessment	Procedural transfer	0.08, ns
Jitendra et al. (2010) *Randomized controlled trial*	Instruction in problem-specific multiple strategies[261] vs. traditional instruction	A total of 29 sessions, each lasting 50 minutes	A total of 472 students in the 7th grade in the United States	Word problems involving numbers and operations	Procedural posttest	0.21**
					Procedural maintenance (one month after posttest)	0.09, ns
					Transfer	−0.01, ns
Perels et al. (2005)[262] *Randomized controlled trial*	Instruction in generic multiple strategies after school[263] vs. no instruction	A total of six weekly sessions, each lasting 90 minutes	Approximately 116 students in the 8th grade in Germany	Word problems involving general math achievement	Procedural posttest	0.46**
Star and Rittle-Johnson (2008) *Randomized controlled trial*[264]	Demonstration of most efficient strategy to solve three different equations vs. additional time to practice solving equations	A total of five sessions, each lasting one hour, conducted on consecutive days during the summer	A total of 66 students in the 6th and 7th grades in the United States	Algebra equations	Procedural posttest	0.02, ns
					Procedural transfer	0.06, ns
					Flexibility posttest (average of three measures)	0.23, ns
			Worked Examples with Students Comparing Strategies			
Rittle-Johnson and Star (2007) *Randomized controlled trial*	Students comparing worked examples solved with multiple strategies vs. students studying worked examples solved with multiple strategies	A total of two sessions, each lasting 45 minutes, across two days	A total of 70 students in the 7th grade in the United States	Algebra equations	Procedural posttest	0.08**
					Conceptual posttest	−0.04, ns
					Flexibility posttest	0.10**
Rittle-Johnson and Star (2009) *Randomized controlled trial*	Students comparing worked examples of one problem solved using multiple strategies vs. students comparing worked examples of equivalent problems solved with the same strategy	Three consecutive class periods	A total of 98 students in the 7th and 8th grades in the United States	Algebra equations	Procedural posttest[265]	−0.14, ns[266]
					Procedural maintenance (two weeks after posttest)	0.01, ns
					Conceptual posttest	0.36*
					Conceptual maintenance (two weeks after posttest)	0.29, ns
					Flexibility posttest (average of two measures)	0.36*
					Flexibility maintenance (two weeks after posttest, average of two measures)	0.50**
Rittle-Johnson et al. (2009) *Randomized controlled trial* Students not using algebra on pretest	Students comparing worked examples of one problem solved using multiple strategies vs. students studying worked examples solved with multiple strategies	A total of three daily sessions, each lasting approximately 45 minutes	55 students in the 7th and 8th grades in the United States	Algebra equations	Procedural posttest	−0.45*[267]
					Conceptual posttest	−0.33*
					Flexibility posttest	−0.35, ns

(continued)

Table D.6. Studies of interventions that involved multiple problem-solving strategies and contribute to the level of evidence rating *(continued)*

Study	Comparison	Duration	Students	Math Content	Domain and Outcome	Effect Size
Rittle-Johnson et al. (2009) *Randomized controlled trial* Students using some algebra on pretest	Students comparing worked examples of one problem solved using multiple strategies vs. students studying worked examples solved with multiple strategies	A total of three daily sessions, each lasting approximately 45 minutes	55 students in the 7th and 8th grades in the United States	Algebra equations	Procedural posttest	0.19, ns[268]
					Conceptual posttest	−0.13, ns
					Flexibility posttest	0.12, ns
Star and Rittle-Johnson (2009a) *Randomized controlled trial*	Students comparing worked examples solved with multiple strategies vs. students studying worked examples solved with multiple strategies	A total of three sessions, each lasting 40 minutes	A total of 157 students in the 5th and 6th grades in the United States	Numbers and operations estimation	Conceptual posttest[269]	−0.06, ns
					Conceptual maintenance (two weeks after posttest)	0.00, ns
					Flexibility posttest	0.43**
					Flexibility maintenance (two weeks after posttest)	0.30*
Students Generating and Sharing Multiple Strategies						
Ainsworth et al. (1998) *Randomized controlled trial*	Students generating multiple strategies vs. no treatment	A total of two sessions, with a total time of 60–90 minutes	A total of 48 students (average age 9) in the United Kingdom	Problem solving involving numbers and operations	Flexibility posttest	1.30**
					Flexibility maintenance (delay after posttest not reported)	1.01**
Ginsburg-Block and Fantuzzo (1998) *Randomized controlled trial*	Students solving a problem with a partner, sharing their strategy and solution with the larger group, and then generating multiple strategies with a partner vs. students solving problems individually without generating or sharing multiple strategies	A total of 14 sessions, each lasting 30 minutes	A total of 52 students in the 3rd and 4th grades in the United States[270]	Word problems involving numbers and operations	Procedural posttest[271]	0.76*[272]
Ginsburg-Block and Fantuzzo (1998) *Additional comparison*[273]	Students solving a problem, sharing their strategy and solutions with the larger group, and then generating multiple strategies individually[274] vs. students solving problems individually without generating or sharing multiple strategies	A total of 14 sessions, each lasting 30 minutes	A total of 52 students in the 3rd and 4th grades in the United States[275]	Word problems involving numbers and operations	Procedural posttest[276]	0.32, ns[277]
Star and Rittle-Johnson (2008) *Randomized controlled trial*[278]	Students being prompted to generate multiple strategies by resolving problems using different ordering of steps vs. students solving similar problems without being prompted to generate multiple strategies	A total of five sessions, each lasting one hour, conducted on consecutive days during the summer	A total of 63 students in the 6th and 7th grades in the United States	Algebra equations	Procedural posttest	−0.35
					Procedural transfer	−0.11, ns
					Flexibility posttest (average of three measures)	0.44*
Star and Seifert (2006) *Randomized controlled trial*	Students were given problems they had previously solved and were asked to re-solve using a different ordering of steps vs. students solving similar problems	A total of three one-hour sessions	A total of 32 students in the 6th grade in the United States	Algebra problems	Procedural posttest	−0.35, ns
					Flexibility posttest	0.43, ns

** = statistically significant at 0.05 level
* = statistically significant at 0.10 level
ns = not statistically significant

Recommendation 5. Help students recognize and articulate mathematical concepts and notation.

Level of evidence: Moderate Evidence

Three studies directly support two suggestions of this recommendation (see Table D.7), and non-robust findings exist for another suggestion; overall, the panel believes a moderate level of evidence supports the full recommendation.[279] The first suggestion, relating problem solving to mathematical concepts, was supported by a study finding that student achievement improved when teachers discussed mathematics problems conceptually (without numbers) and then represented them visually before focusing on the mathematical operations and notation.[280] Two other studies that tested the impact of teaching algebra notation to students, another suggestion, found positive effects.[281] Finally, three studies examined student self-explanation, the second suggestion. The results were inconsistent across the studies, with two studies reporting positive effects and one reporting no discernible effects.[282]

Relating mathematics to problem solving. One study meeting WWC standards found that discussing problem structure and visual representation prior to formulating and computing math problems had positive effects on student achievement.[283] In the intervention, teachers discussed word problems with 4th-grade students without using numbers, to encourage students to think about problem structure and apply their informal mathematical knowledge. Students then visually represented the problems; teachers also modeled the representation and discussed it with students. Only after these conceptual processes did students write a number sentence and solve the problem. Comparison students received practice worksheets with the same problems. Intervention students scored higher than comparison students on multiplication and division word problems, and this positive effect persisted for at least two months after the intervention ended.[284]

Student explanation. Three studies used worked examples to examine student self-explanation of the solution process. Results were not robust, with two studies finding positive results and one study finding no discernible effects.[285] These four studies had diverse student samples and were conducted in four different countries, with students ranging from 4th to 9th grade.[286] The mathematical content also was varied, ranging from numbers and operations to algebra.[287] The intervention details varied as well.

In one study, teachers had students complete four in-class assignments.[288] For intervention students, these assignments included 5 to 6 worked examples, some solved correctly and some solved incorrectly, and students were asked to explain why the solutions were correct or incorrect. Intervention students also received 5 to 6 practice problems to solve. Comparison students received 10 to 12 practice problems only. The authors found positive effects on the conceptual knowledge posttest.

In the two other studies, all students received worked examples, but only students in the intervention group were asked to explain each step in the process. Intervention students in the first of these were asked to self-explain each step in the problem-solving process. These students were able to solve significantly more word problems than students who were asked to review only the problems and learn each step.[289] This positive effect persisted for one month after the intervention ended. Intervention students in the second study were asked to pretend they were explaining each step in the worked examples to another student; students in the comparison condition were asked to study the worked examples until they understood how the problems were solved.[290] This intervention had no discernible effects.

Algebraic problem solving. Two studies meeting WWC standards directly tested the effect of helping students make sense of algebraic notation. The first study used an algebra tutoring program to change the order of steps high school students took to solve algebra problems—a slight change that had a statistically significant effect on later achievement.[291] The authors proposed that solving intermediate arithmetic problems before representing them with algebraic notation helps students understand problem structure using the mathematical knowledge (arithmetic) they already possess, and students can then use this existing knowledge to more easily determine algebraic notation.[292] In the intervention condition, students were asked to solve two intermediate arithmetic problems before providing algebraic notation, while in the comparison condition students were asked to provide algebraic notation first. Students in both groups then solved new algebra word problems, and the authors reported that the intervention students solved more problems correctly.[293]

The second study examined two-step word problems in which students had to substitute one algebraic expression into another.[294] Students in the intervention condition were given four symbolic problems that required them to substitute one expression into another (for example, "Substitute $62 - f$ for b in $62 + b$."). Comparison students were asked to solve four one-step word problems; these word problems were very similar in format to one step of the two-step word problems that were the outcomes. The authors found that students in the intervention group correctly solved more two-step word problems.

Table D.7. Studies of interventions that helped students recognize and articulate concepts and contribute to the level of evidence rating

Study	Comparison	Duration	Students	Math Content	Outcome[295]	Effect Size
Relating to Conceptual Understanding						
Huinker (1992) *Randomized controlled trial*	Student and teacher discussion of problem representation and connection to a mathematical operation prior to formal mathematics[296] vs. solving practice problems	A total of 18 lessons	A total of 128 students in the 4th grade in the United States	Word problems involving numbers and operations	Posttest[297]	1.21**
					Retention (two to three months after posttest)	1.24**
Student Explanation of Worked Examples						
Booth et al. (2010) *Randomized controlled trial*	Students presented with worked examples and asked to explain why the solution was correct or incorrect vs. students given similar problems to solve	Four sessions	A total of 51 high school students in the United States[298]	Algebra	Conceptual posttest[299]	0.50*
Mwangi and Sweller (1998)[300] *Randomized controlled trial*	Students asked to explain each step in worked examples vs. students asked to study worked examples until they understand the solution	One session	A total of 48 students in the 4th grade in Australia	Word problems involving numbers and operations	Posttest	0.00, ns
					Retention (10 days after posttest)	−0.21, ns
					Transfer (10 days after posttest)	−0.11, ns
Tajika et al. (2007) *Quasi-experimental design*	Students asked to explain each step in worked examples vs. students provided with explanations for each step in worked examples and told to study them	One 20-minute session	A total of 53 students in the 6th grade in Japan	Word problems involving numbers and operations	Posttest (one week after intervention)	0.93**
					Transfer (one month after posttest)	0.58**
Making Sense of Algebra Notation						
Koedinger and Anderson (1998) *Randomized controlled trial*	Students asked to solve related arithmetic questions before being asked to represent the problem algebraically vs. students asked to represent problems algebraically before solving related arithmetic questions	Two sessions, each lasting one to two hours	A total of 20 high school students in the United States[301]	Word problems involving general math achievement (algebra and numbers and operations)	Posttest	Not reported ($p < 0.05$)
Koedinger and McLaughlin (2010) *Randomized controlled trial*	Students given practice word problems that involved substituting one algebraic expression into another vs. students given practice word problems that did not involve substitution	One session	A total of 303 middle school students in the United States	Word problems involving algebra	Posttest[302]	0.26**

** = statistically significant at 0.05 level
* = statistically significant at 0.10 level
ns = not statistically significant

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